LEONHARDI EULERI OPERA OM

SUB AUSPICIIS SOCIETATIS SCIENTIARUM NATURALIUM HELV

EDENDA CURAVERUNT

FERDINAND RUDIO · ADOLF KRAZER · PAUL STÄCKEL

SERIES I · OPERA MATHEMATICA · VOLUMEN XX

LEONHARDI EULERI

COMMENTATIONES ANALYTIC

AD THEORIAM INTEGRALIUM ELLIPTICORUM PERTINENTES

EDIDIT

ADOLF KRAZER

VOLUMEN PRIUS

LIPSIAE ET BEROLINI

TYPIS ET IN AEDIBUS B. G. TEUBNERI

MCMXII

LEONHARDI EULERI

OPERA OMNIA

LEONHARDI EUL

OPERA OMNIA

SUB AUSPICIIS
SOCIETATIS SCIENTIARUM NATURA
HELVETICAE

EDENDA CURAVERUNT

FERDINAND RUDIO
ADOLF KRAZER PAUL STÄCKEL

SERIES PRIMA
OPERA MATHEMATICA
VOLUMEN VICESIMUM

LIPSIAE ET BEROLINI
TYPIS ET IN AEDIBUS B. G. TEUBNERI
MCMXII

LEONHARDI EULERI

MMENTATIONES ANALYTICAI

AD THEORIAM INTEGRALIUM ELLIPTICORUM PERTINENTES

EDIDIT

ADOLF KRAZER

VOLUMEN PRIUS

LIPSIAE ET BEROLINI
TYPIS ET IN AEDIBUS B. G. TEUBNERI
MCMXII

VORWORT DES HERAUSGEBERS

In den 20. und 21. Band der I. Serie von *Leonhardi Euleri Opera omni*
Abhandlungen Eulers aufgenommen worden, welche sich mit Integralen beschä
r elliptische nennen, weil das zur Rektifikation der Ellipse dienende zu ihnen
Dieses Integral zog dadurch, daß es sich nicht durch die bekannten Funktione
ließ, die Aufmerksamkeit der Mathematiker seit dem Ausgange des 17. Jahrhu
em Maße auf sich und so sehen wir auch Euler bald nach Beginn seiner mathema
ahn, 1733, mit ihm beschäftigt. In der Abhandlung 28 (des Eneströmsche
isses), mit der der vorliegende Band beginnt, findet Euler, daß mit Hilfe der
n der Ellipse die Lösung einer gewissen Differentialgleichung erster Ordnun
e Trennung der Variablen nicht möglich ist, konstruiert werden könne. Der nä
ke, die Rektifikation der Ellipse zur Lösung von Differentialgleichungen zu verw
ftigt ihn auch 1734 in der folgenden Abhandlung 52 und führt hier zur Lösu
be, auf einer Schar von Ellipsen mit gleicher einen Achse und gemeinsamem S
e von diesem aus gleichlange Bogenstücke abzuschneiden. Nach dieser Abha
eine lange Pause ein und wir sehen erst 1749 Euler wieder mit dem Rektifik
an der Ellipse beschäftigt; in 154 gibt er eine Reihenentwickelung für den
llipse.

Die geringe Zahl dieser Abhandlungen und auch die Art ihrer Problemstellu
nen, daß Euler zu einer fruchtbaren Entwickelung seiner Untersuchungen übe
Integrale einer Anregung von außen bedurfte, und wir wissen auch, wann u
m geworden ist. Am 23. Dezember 1751 wurde er von der Berliner Akade
gt, die ihr von Fagnano übersandten *Produzioni* zu prüfen, ehe man dem V
rte, und schon am 27. Januar 1752 liest Euler in der Akademie eine Abha
in welcher er für die auf die Ellipse und Hyperbel bezüglichen Resultate Fa
einfachere Ableitung gibt, die auf die Lemniskate bezüglichen aber wesentlich er
r erfaßt sogleich die Bedeutung dieser Untersuchungen für die Integralrechnu

ohne aber auf ihren Inhalt, von dem er damals wohl noch gar nicht Kenntn
nen hatte, einzugehen) erfuhr, daß LAGRANGE schon vor geraumer Zeit im 4. Band
cellanea Taurinensia eine direkte Methode zur Integration seiner Differentialgleich
teilt habe. Zwar hatte auch er ungefähr um dieselbe Zeit, 1765, in 345 eine
geben, war aber wegen der Umständlichkeit der benutzten Substitutionen nicht
edigt. Jetzt bemächtigte er sich in 506 und in einer weiteren, noch im selben Jahre
ßten Abhandlung 676 der Methode von LAGRANGE und benutzte sie zur Integ
von ihm früher behandelten Differentialgleichungen.

Eine zweite Gruppe von Arbeiten EULERS über elliptische Integrale wurde dur
die Mitte des 18. Jahrhunderts erschienenen Abhandlungen von MACLAURIN und D'A
r veranlaßt, von denen der erstere mit geometrischen, der letztere mit analyti
smitteln eine Anzahl von Integralen abgeleitet hatte, die sich durch einfache Sub
n auf die Rektifikation der Ellipse und Hyperbel reduzieren lassen. An diese Ab
en knüpfte EULER an; die früheste durch sie hervorgerufene Abhandlung Eu
die aus dem Jahre 1759 stammende 295, auf die erst im nächsten Jahre
e. Beide beschäftigen sich mit den Integralen $\int\sqrt{\frac{f+gx^2}{h+kx^2}}\,dx$ und teilen die
Klassen, je nachdem das Integral durch einen einzigen Kegelschnittbogen, durch
en und eine algebraische Funktion, oder endlich durch zwei Kegelschnittbogen un
raische Funktion ausgedrückt wird. Es liegen hier die Keime der Reduktion
tischen Integrale auf eine Normalform vor, sie kommen aber wegen des Überwie
geometrischen Vorstellungen nicht zur Entfaltung. EULER verschärfte diese
ung noch, indem er in der Folge nach Kurven suchte, deren Bogenelement
passende Substitution in das einer Ellipse übergehe, und so die Übereinstimmun
grale ohne Hinzutritt einer algebraischen Funktion verlangte. Eingeleitet wurden
n Untersuchungen durch die Abhandlung 590 des Jahres 1775. Diese gibt drei
an. Das erste, daß alle imaginären Größen, die „in calculo analytico" auft
in die Form $a+bi$ gebracht werden können, gehört nicht hierher; das zweite
daß es außer dem Kreise selbst keine algebraische Kurve gebe, deren Bogen
sbogen allein, und das dritte, daß es keine solche Kurve gebe, deren Bogen durch
arithmus allein dargestellt werden können. EULER fordert die Mathematiker au
Theoreme strenge Beweise zu liefern. Er zeigt dann 1776, wie man im Gege
zu einer gegebenen Parabel (638), zu einer gegebenen Ellipse (639) und zu
ve, deren Bogendifferential von der Form $\frac{v^{m-1}\,dv}{\sqrt{1-v^{2n}}}$ (645) ist, unendlich viele
che Kurven angeben könne, die das gleiche Bogendifferential besitzen, und unter
33 die allgemeinen Bedingungen, unter denen die Bogendifferentiale zweier K
einstimmen. Für die in 638, 639, 633 behandelten Probleme gab EULER später,

in 781, 780, 782 neuerdings Lösungen und bei dieser Gelegenheit
früher in 590 für den Kreis aufgestellte Theorem nicht richtig sei.
mehr unendlich viele algebraische Kurven, die keine Kreise sind, ang
differential dem eines gegebenen Kreises gleich ist. In der den *Opera*
Abhandlung 817 wird das in Rede stehende Problem noch einmal und zw
die Parabel und die Ellipse gelöst.

Vier Abhandlungen, die in den Bänden 20 und 21 Platz gefun
nicht genannt. Alle vier haben das Gemeinsame, daß Reihenentwicklu
lichsten Inhalt bilden. Abhandlung 448 nimmt das schon in 154 be
Reihenentwicklung für den Ellipsenumfang wieder auf, 605 behandelt
elastischen Kurve, 624 gibt Reihenentwicklungen für die Oberfläche
und 819 solche für den Hyperbelbogen.

Unter den Manuskripten, die die Petersburger Akademie der Re
gestellt hat, befinden sich die Originale der Abhandlungen 817 und 818
von 28, 251, 252, 261, 263 und 264. Diese Manuskripte stimmen mit d
überein; nur das Manuskript des Summariums von 28 ist bisher noch
wesen und erscheint hier zum ersten Male (am Schlusse des Bandes 20, d
gedruckt war, als das Manuskript vorgefunden wurde).

Wenn man den Gesamtinhalt der EULERSCHEN Abhandlungen üb
und ihre Bedeutung für die spätere Entwicklung der Theorie derselbe
in dem einen Teile dieser Arbeiten (insbesondere 252, 261, 581) niederg
Additionstheoreme der Integrale EULERS gewaltiges und bleibendes V
aber, warum das in dem anderen Hauptteil der Abhandlungen (insbe
handelte Problem der Reduktion der Integrale auf feste Normalfor
führung des allgemeinen Integrals auf diese, trotzdem es für EULE
und wollte wie kein anderer, wie geschaffen war, keine so glückliche
so müssen wir, wie schon oben erwähnt, dem Nichtloskommen vo
stellungen die Schuld geben. Ein entscheidender Fortschritt in di
erst geschehen, wenn die geometrische Grundlage, der allerdings die
tischen Integralen bisher fast alles verdankte, zurücktrat und einer Be
um ihrer selbst willen Platz machte; der dies leisten sollte, war sch
EULER 1783 die Augen schloß: LEGENDRE.

Karlsruhe, den 1. November 1912.

A

INDEX

Insunt in hoc volumine indicis Eneströmiani commentationes
28, 52, 154, 211, 251, 252, 263, 261, 264, 278, 295, 345, 347, 448

SPECIMEN DE CONSTRUCTIONE AEQUATIONUM DIFFERENTIALIUM [SI]NE INDETERMINATARUM SEPARATIONE

Commentatio 28 indicis ENESTROEMIANI
[Comm]entarii academiae scientiarum Petropolitanae 6 (1732/3), 1738, p. 168—174

[In]determinatarum separationem in aequationibus differentialibus ideo [...]to desiderari, quod ex ea inventa aequationis constructio sponte [...]e in his rebus exercitato satis perspectum esse arbitror. Integratio [a]equationum differentialium, siquidem succedit, optime indeterminatis [...] instituitur. Quanquam enim innumerabiles dantur aequationes, [in]tegrales sine huiusmodi separatione inveniri possunt, cuiusmodi [...] exhibuit Celeb. IOH. BERNOULLI in Comm. nostrorum Tom. I [...] tamen eae aequationes omnes ita sunt comparatae, ut vel per se [i]ndeterminatarum separatio, vel saltem ex ipsa integratione facile [...] Similis vero est etiam ratio constructionum, quibus adhuc usi [...]stae; sunt enim omnes huiusmodi, ut aequationis, si nullo alio modo [...]tae a se invicem separari possunt, separatio tamen ex ipsa con[...] proficiscatur. Hanc ob rem nullam adhuc exhiberi posse existimo [...]n differentialem construibilem, cuius separatio omnes vires eluderet.

[...]per[2]) autem in ellipsi rectificanda occupatus inopinato incidi in ae[...] differentialem, quam ope rectificationis ellipsis construere poteram,

[1]) [I.] BERNOULLI, *De integrationibus aequationum differentialium, ubi traditur methodi alicuius [...]randi sine praevia separatione indeterminatarum*, Comment. acad. sc. Petrop. 1 [...], p. 167; *Opera omnia* T. 3, p. 108. A. K.

[2]) [E]ULERI Commentatio 11 (indicis ENESTROEMIANI): *Constructio aequationum quarundam [...], quae indeterminatarum separationem non admittunt*, Nova acta erud. 1733, [...] *[LEON]HARDI EULERI Opera omnia*, series I, vol. 22. A. K.

neque tamen indeterminatarum separatio nequidem ex ipse
inveniri poterit. Aequatio vero, quam obtinui, erat haec

$$dy + \frac{y^2 dx}{x} = \frac{x dx}{x^2 - 1},$$

Riccatianae fere similis et forte ad separandum aeque
$dy + y^2 dx = x^2 dx$. Casus hic mihi primum vehementer pa
at constructione attentius perspecta facile intellexi ex ea
tionem indeterminatarum non posse deduci, sed etiam, si a
haec succederet, multo maiora secutura esse absurda, com
perimetrorum ellipsium dissimilium, quae, ut mihi quider
analysin superat. Constructio autem ipsa perquam est faci
elongatione infinitarum ellipsium alterutrum axem commu
hanc ob rem consueto per quadraturas construendi modo lon

3. Proponam igitur totam rem, prout ad eam perveni.
quadrans ellipticus, cuius centrum C, semi-axes vero AC

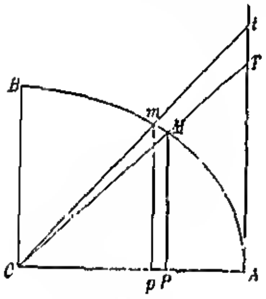


Fig. 1.

$AC = a$ et $BC = b$ et ex A ducat
nita AT ad eamque ex centro C
CT abscindens arcum $AM = s$ v
Demisso ex M in AC perpendicul
erit ex natura ellipsis $PM = \frac{b\sqrt{}}{}$
analogiam $CP : PM = CA : AT$ ha

$$tx = b\sqrt{(a^2 - x^2)} \quad \text{seu} \quad x =$$

Sumatur arcus AM elementum Mm ducanturque mp, Ct p
proximae; erit

$$Mm = ds = \frac{-dx\sqrt{(a^4 - (a^2 - b^2)x^2)}}{a\sqrt{(a^2 - x^2)}}$$

et $Tt = dt$. Quia autem est $x = \frac{ab}{\sqrt{(b^2 + t^2)}}$, erit d
$\sqrt{(a^2 - x^2)} = \frac{at}{\sqrt{(b^2 + t^2)}}$ et $\sqrt{(a^4 - (a^2 - b^2)x^2)} = \frac{a\sqrt{(b^4 + a^2tt)}}{\sqrt{(b^2 + t^2)}}$.

$$ds = \frac{bdt\sqrt{(b^4 + a^2tt)}}{(bb + tt)^{\frac{3}{2}}}.$$

Ad cuius integrale per seriem saltem inveniendum pono $a^2 = (n+1)$
prodeat

$$ds = \frac{b^2 dt \sqrt{((b^2+t^2)+nt^2)}}{(b^2+t^2)^{\frac{3}{2}}},$$

superiusque irrationale fit binomium, cuius alterum membrum est
alterumque simplex terminus nt^2. Resolvo nunc $\sqrt{((b^2+t^2)+nt^2)}$ per c
notum in seriem hanc

$$(b^2+t^2)^{\frac{1}{2}} + \frac{Ant^2}{(b^2+t^2)^{\frac{1}{2}}} + \frac{Bn^2t^4}{(b^2+t^2)^{\frac{3}{2}}} + \frac{Cn^3t^6}{(b^2+t^2)^{\frac{5}{2}}} + \text{etc.},$$

in qua brevitatis gratia est

$$A = \frac{1}{2}, \quad B = -\frac{1\cdot 1}{2\cdot 4}, \quad C = \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}, \quad D = -\frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8} \text{ etc.}$$

Habebitur ergo

$$ds = \frac{b^2 dt}{b^2+t^2} + \frac{Ab^2nt^2dt}{(b^2+t^2)^2} + \frac{Bb^2n^2t^4dt}{(b^2+t^2)^3} + \frac{Cb^2n^3t^6dt}{(b^2+t^2)^4} + \text{etc.}$$

et integer arcus ellipticus s erit integrale huius seriei.

4. Notandum hic est singulorum horum terminorum integration
primi termini $\int \frac{bbdt}{bb+tt}$ posse reduci; dat vero $\int \frac{bbdt}{bb+tt}$ arcum circuli
cuius tangens est t. Hanc ob rem singulos terminos assumto hoc c
arcu integrabo, ut sequitur:

$$\int \frac{b^2t^2dt}{(b^2+t^2)^2} = \frac{1}{2}\int \frac{bbdt}{bb+tt} - \frac{1}{2}\frac{b^2t}{bb+tt},$$

$$\int \frac{b^2t^4dt}{(b^2+t^2)^3} = \frac{1\cdot 3}{2\cdot 4}\int \frac{b^2dt}{bb+tt} - \frac{1\cdot 3}{2\cdot 4}\frac{b^2t}{bb+tt} - \frac{1}{4}\frac{b^2t^3}{(bb+tt)^2},$$

$$\int \frac{b^2t^6dt}{(b^2+t^2)^4} = \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\int \frac{b^2dt}{bb+tt} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{b^2t}{bb+tt} - \frac{1\cdot 5}{4\cdot 6}\frac{b^2t^3}{(bb+tt)^2} - \frac{1}{6}\frac{b^2t}{(bb+}$$

ex quibus lex integralium reliquorum terminorum iam satis apparet.

5. Si quarta perimetri ellipticae pars AMB requiratur, oportet
infinitum hocque facto omnes termini algebraici in superioribus inte

evanescunt. Arcus circularis vero $\int\frac{bbdt}{bb+tt}$ posito $t=\infty$
pheriae circuli partem, cuius radius est b seu BC,
littera e. Erit propterea

$$\int\frac{b^2dt}{bb+tt}=e,\quad \int\frac{b^2t^2dt}{(bb+tt)^2}=\frac{1\cdot e}{2},$$

$$\int\frac{b^2t^4dt}{(bb+tt)^3}=\frac{1\cdot 3\cdot e}{2\cdot 4},\quad \int\frac{b^2t^6dt}{(bb+tt)^4}=\frac{1\cdot 3\cdot 5\cdot e}{2\cdot 4\cdot 6}$$

Prodibit igitur quarta perimetri ellipticae pars

$$AMB=e\Big(1+\frac{1}{2}An+\frac{1\cdot 3}{2\cdot 4}Bn^2+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}Cn^3+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}$$

Atque substitutis loco A, B, C, D etc. valoribus debitis

$$AMB=e\Big(1+\frac{1\cdot n}{2\cdot 2}-\frac{1\cdot 1\cdot 3\cdot n^2}{2\cdot 2\cdot 4\cdot 4}+\frac{1\cdot 1\cdot 3\cdot 3\cdot 5\cdot n^3}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}-\frac{1\cdot 1\cdot 3\cdot}{2\cdot 2\cdot 4\cdot}$$

6. Haec series, si n est valde parvum seu $\frac{a^2-b^2}{b^2}$, id
ellipsis admodum propinqua est circulo, vehementer con
igitur facile ellipsis perimeter invenitur. Quando vero n
minima seu $a=b+\omega$ denotante ω quantitatem quam mi
et $AMB=e\left(1+\frac{\omega}{2b}\right)$ quam proxime. Quando vero fit $a=$
A in C et evadit $AMB=BC=b$; hoc vero casu erit
igitur

$$\frac{b}{e}=1-\frac{1}{2\cdot 2}-\frac{1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}-\frac{1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}-$$

Summa huius seriei ergo exprimit rationem radii ad
partem in circulo.

7. Quemcumque igitur habeat valorem littera n in serie
seriei semper poterit assignari ope rectificationis ellipsis,
d minorem ut $\sqrt{(n+1)}$ ad 1. Hoc cum ita sc
ethodo mea summationes seriorum ad resolut
quam nuper[1]) exhibui, ut investigarem, a cui

Commentatio 25 (indicis ENESTROEMIANI): *Methodus*
t. acad. sc. Petrop. 6 (1732/3), 1738, p. 68;
14 A. K.

mmatio inventae seriei pendeat. Quo autem haec methodus facilius iberi, pono $n = -x^2$ eritque summanda ista series

$$1 - \frac{1 \cdot x^2}{2 \cdot 2} - \frac{1 \cdot 1 \cdot 3 \cdot x^4}{2 \cdot 2 \cdot 4 \cdot 4} - \frac{1 \cdot 1 \cdot 3 \cdot 3 \cdot 5 \cdot x^6}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} - \text{etc.};$$

ır summam pono s. Erit ergo differentiando

$$\frac{ds}{dx} = -\frac{1 \cdot x}{2} - \frac{1 \cdot 1 \cdot 3 \cdot x^3}{2 \cdot 2 \cdot 4} - \frac{1 \cdot 1 \cdot 3 \cdot 3 \cdot 5 \cdot x^5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6} - \text{etc.}$$

ɔ per x multiplico sumoque differentialia posito dx constante; erit

$$\frac{d.xds}{dx^2} = -1 \cdot x - \frac{1 \cdot 1 \cdot 3 \cdot x^3}{2 \cdot 2} - \frac{1 \cdot 1 \cdot 3 \cdot 3 \cdot 5 \cdot x^5}{2 \cdot 2 \cdot 4 \cdot 4} - \text{etc.}$$

do ubique per x contraque per dx multiplico sumoque integralia; erit

$$\int \frac{d.xds}{xdx} = -x - \frac{1 \cdot 1 \cdot x^3}{2 \cdot 2} - \frac{1 \cdot 1 \cdot 3 \cdot 3 \cdot x^5}{2 \cdot 2 \cdot 4 \cdot 4} - \text{etc.}$$

terum per dx multiplico, divido vero per x^3 et sumo integralia; erit

$$\int \frac{1}{x^3} \int \frac{d.xds}{x} = \frac{1}{x} - \frac{1 \cdot x}{2 \cdot 2} - \frac{1 \cdot 1 \cdot 3 \cdot x^3}{2 \cdot 2 \cdot 4 \cdot 4} - \frac{1 \cdot 1 \cdot 3 \cdot 3 \cdot 5 \cdot x^5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} - \text{etc.}$$

ɔ series est ipsa initialis per x divisa; eius igitur summa est $\frac{s}{x}$. habemus hanc aequationem

$$\int \frac{1}{x^3} \int \frac{d.xds}{x} = \frac{s}{x},$$

tis differentialibus abit in hanc

$$x^2 ds - sxdx = \int \frac{d.xds}{x}.$$

stur haec denuo; prodibit

$$x^2 dds + xdxds - sdx^2 = \frac{d.xds}{x} = dds + \frac{dxds}{x}.$$

quationis resolutio igitur pendet a summatione seriei propositae; quae rectificationem ellipsis habeatur, aequationis constructio quoque

8. Cum in ista aequatione s ubique unam teneat d
ea poterit per methodum meam Tom. III Comm.[1]) inserta
simpliciter differentialem facta substitutione $s = c^{\int p dx}$, ubi
cuius log. est 1. Hoc posito erit $ds = c^{\int p dx} p dx$ et $dds = c$
atque aequatio inventa transformabitur in hanc

$$x^2 dp + x^2 p^2 dx + pxdx - dx = dp + ppdx +$$

quae divisa per $xx - 1$ mutatur in istam

$$dp + ppdx + \frac{pdx}{x} = \frac{dx}{xx - 1}.$$

Ad hanc simpliciorem efficiendam pono $p = \frac{y}{x}$ et provenie

$$dy + \frac{yydx}{x} = \frac{xdx}{xx - 1}.$$

Quae quomodo separari possit, neque perspicio neque
sideratio eo perducit.

9. Quo autem ipsa constructio huius aequationis ex pr
catur, pono illum axis semissem AC, quem ante littera
lem r, quia ut variabilis debet considerari, et quartam
partem respondentem q; erit $-xx = n = \frac{r^2 - b^2}{b^2}$ et $x = V($
$q = es$; est vero $s = c^{\int p dx} = c^{\int \frac{y dx}{x}}$, quocirca habebitur $q = ec^{\int y}$
adeoque $y = \frac{xdq}{qdx} = \frac{(r^2 - b^2) dq}{qrdr}$. Ne autem, quando r maior e
nalia proveniant, restituo loco xx valorem $-n$; erit $\frac{dx}{x} = \frac{dn}{2n}$
His substitutis habebitur ista aequatio

$$2dy + \frac{y^2 dn}{n} = \frac{dn}{n + 1},$$

1 Commentatio 10 (indicis ENESTROEMIANI): *Nova methodi*
secundi gradus reducendi ad aequationes differentiales pri
1 (1728), 1732, p. 124; LEONHARDI EULERI *Opera om*

etur sumendis $n = \frac{r^2 - b^2}{b^2}$ et $y = \frac{(r^2 - b^2)dq}{qrdr}$ seu, iam invento n, hinc sequens nascitur constructio:

o quadrante elliptico BCA (Fig. 2), cuius centrum in C et semi-stans est, puta $= 1$, pono hic 1 loco b, quo facilius homogeneitas ri. Erit ergo semi-axis $AC = r$; ex A erigatur normalis elliptico AB; erit punctum D in curva aliqua BD, cuius con- modo est in promtu. In ea igitur q. Sit F huius ellipsis focus; erit 1); et ad BF ducatur normalis FP; $- 1 = n$. Notetur hic, quando fit focus F in BC incidit, valorem n um et ex altera parte puncti C versus ortere. Deinceps ducatur tangens DT in D; erit

$$AT = \frac{qdr}{dq};$$

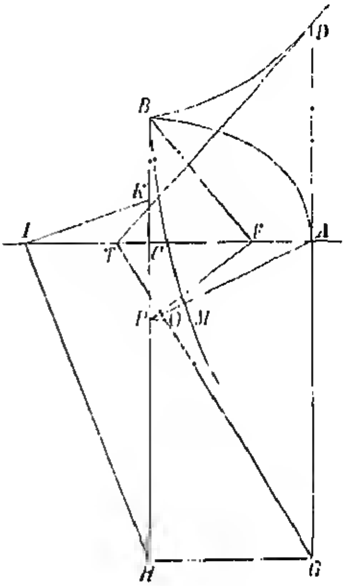


Fig. 2.

P ex T ducatur recta TG normaliter si opus est, productam in O et DA currens in G; erit ob similia triangula G

$$AG = \frac{rqdr}{(r^2 - 1)dq}.$$

ualis capiatur CH et sumta $CI = CB = 1$ ad ductam HI erigatur ris IK; erit

$$CK = \frac{(r^2 - 1)dq}{rqdr} = y.$$

it aequalis PM eritque M in curva quaesita BM; huius enim est proprietas, ut dictis $CP = n$ et $PM = y$ sit

$$2dy + \frac{y^2 dn}{n} = \frac{dn}{n+1}.$$

SOLUTIO PROBLEMATUM
RECTIFICATIONEM ELLIPSIS REQU

Commentatio 52 indicis ENESTROEMIANI
Commentarii academiae scientiarum Petropolitanae 8 (1736),

1. Agitata iam superiori seculo inter Geometras sunt
in quibus linea curva requirebatur, quae ab infinitis
arcus aequales abscinderet. Communicaverunt etiam ill
metrae[1]) elegantes solutiones pro casu, quo curvae po
sunt similes, uti cum ab infinitis circulis vel paraboli
scindendi essent. Nemo autem, quantum constat, ulteri
que quisquam pro curvis dissimilibus problemati satisf
quaestio de infinitis ellipsibus proponeretur. Atque et
Geometrae per litteras significassem[2]) me aequationem
infinitis ellipsibus dissimilibus arcus aequales abscindere
respondit huius problematis solutionem in sua non
simul rogavit, ut meam solutionem in non contemnend
tum communicarem.

2. Huius autem quaestionis summa difficultas i
diversarum et dissimilium ellipsium rectificationes a se
Hanc enim ob causam curvae ab infinitis ellipsibus arcu

1) ... BERNOULLI, *Solutio sex problematum fraternorum in I*
Probl. 4 et 5), Acta erud. 1698 p. 226; *Opera* p. 796;
1, p. 256. A. K.

2) ... ad DAN. BERNOULLI, Novembri (?) 1734; vide
LEONHARD EULER *und* DANIEL BERNOULLI, Biblio
140. A. K.

uationem inventu maxime difficilem esse oportet, eo quod etiam con
us ellipsis rectificatione reliquarum tamen omnium rectificatio ab ista
deat. Deinde methodus, qua in huiusmodi problematis uti solent, it
nparata, ut tantum ad curvas similes accommodari possit, pro curvis
ilibus *autem nullam afferat utilitatem.*

3. Quod autem mihi primum viam ad huiusmodi difficilia proble
efecit, est praecipue universalis mea series summandi methodus.[1]) Hac
enta statim[2]) aequationem differentialem, in qua indeterminatae nullo
e invicem separari possunt, ope rectificationis ellipsium dissimilium
uxi atque paulo post[3]) maxime agitatae aequationis RICCATIANAE cons
nem et resolutionem communicavi.

4. Postmodum autem, cum haec per series operandi methodus
rosa et non satis genuina videretur, in aliam magis naturalem meth
huius modi quaestionibus magis accommodatam inquisivi; atque ta
voto obtinui, ita ut eius beneficio non solum priora problemata,
orum ope resolveram, sed etiam innumera alia, ad quae tractanda
sufficiunt, perficere potuerim. Methodum etiam hanc fuse exposi
sertatione *De infinitis curvis eiusdem generis*[4]) anno praecedente [1734]
ita; quia vero, ne nimis essem prolixus, nulla adieci exempla, non
aret, quam late ea pateat quamque amplum in re analytica a
npum.

5. Quo igitur huius methodi vis et utilitas melius percipiatur, hac d
one eam ad infinitas ellipses accommodabo atque non solum monst

1) Vide notam 1 p. 4. A. K.

2) L. EULERI Commentatio 28 (indicis ENESTROEMIANI); vide p. 1. A. K.

3) L. EULERI Commentatio 31 (indicis ENESTROEMIANI): *Constructio aequationis differe*
$dx = dy + y^2dx$, Comment. acad. sc. Petrop. 6 (1732/3), 1738, p. 231; *LEONHARDI*
a omnia, series I, vol. 22. A. K.

4) L. EULERI Commentatio 44 (indicis ENESTROEMIANI): *De infinitis curvis eiusdem g*
methodus inveniendi aequationes pro infinitis curvis eiusdem generis, Comment. aca
rop. 7 (1734/5), 1740, p. 174; *LEONHARDI EULERI Opera omnia*, series I, vol. 22.

quomodo ab infinitis ellipsibus arcus aequales abscindi
innumerabilium tam primi quam secundi gradus aequatio
resolutionem ope rectificationis ellipsium perficere docebo

6. Quod enim ad curvam, quae ab infinitis ellips
abscindat, attinet, eius constructio eo ipso est facilis, quo
curvarum, quae facillime describi possunt, perfici queat.
constructionem longe anteferendam esse censeo aliis per q
peractis constructionibus. Non igitur tam illius curvae c
quam eius aequatio, quo, quales aequationes tam faci
cognoscatur. Hanc ob rem analysis non parum augmen
aequationes proferantur, quae ope rectificationis ellipsium
mittunt.

7. Considero igitur *primum* infinitas ellipses AM
omnes alterum axem, cuius semissis est CD, habeant
transversos AB diverso
his omnibus ellipsibus
sint abscindendi vel i
aequales, vel curva si
constructio ope harum
cunque praescribitur, u

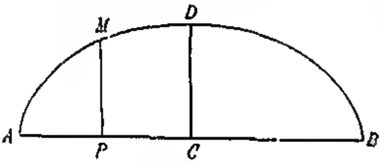


Fig. 1.

omnia solvenda opus est, ut aequatio habeatur inter arc
AP et axem AB, in qua hae tres quantitates insint tan

8. Huiusmodi ergo problematum solutio perficietur, s
modularis, quemadmodum in citata dissertatione de curv
generis docui, inter arcum AM et abscissam AP et axem
bilem. Quo igitur ad huiusmodi aequationem modularem
abscissam $AP = t$, applicatam $PM = u$, arcum $AM = z$, s
$AC = a$, semiaxem constantem $CD = c$. His vero positis e
seu posito $t = ax$ erit $u = c\sqrt{(2x - xx)}$ atque $dt = adx$
Ex his igitur fiet

$$dz = \frac{dx\sqrt{(2a^2x - a^2x^2 + c^2 - 2c^2x + c^2x^2)}}{\sqrt{(2x - xx)}}$$

$-c^2 = b^2$ erit

$$z = \int \frac{dx\sqrt{[c^2 + b^2(2x - xx)]}}{\sqrt{(2x - xx)}}.$$

grali huic invento aequatur ergo z, si integratio fiat posito tan-
ili, b vero et c constantibus. Praeterea in integratione talis addi
ns, ut evanescat z posito $x = 0$. At quia aequatio desideratur,
n eius loco b aeque tanquam variabilis insit ac x et z, quaeritur
erentialis, quae proditura esset, si

$$\int \frac{dx\sqrt{[c^2 + b^2(2x - xx)]}}{\sqrt{(2x - xx)}}$$

entietur posito praeter x etiam b variabili.

atur nunc secundum methodum anno praeterito traditam x con-
erentietur quantitas $\frac{\sqrt{[cc + bb(2x - xx)]}}{\sqrt{(2x - xx)}}$; prodibit $\frac{bdb\sqrt{(2x - xx)}}{\sqrt{[cc + bb(2x - xx)]}}$.
posito quoque b variabili erit

$$dz = \frac{dx\sqrt{[cc + bb(2x - xx)]}}{\sqrt{(2x - xx)}} + db\int \frac{bdx\sqrt{(2x - xx)}}{\sqrt{[cc + bb(2x - xx)]}},$$

num integrale ita debet accipi, ut evanescat posito $x = 0$; in eo
b tanquam constans inest. Ponatur brevitatis gratia

$$R = \frac{dz}{db} - \frac{dx\sqrt{[cc + bb(2x - xx)]}}{db\sqrt{(2x - xx)}};$$

$$R = \int \frac{bdx\sqrt{(2x - xx)}}{\sqrt{[cc + bb(2x - xx)]}}.$$

nunc integrale, cui R aequatur, reduci posset ad integrationem
i z aequalis est, pro R inveniri posset valor finitus per z, qui
in altera aequatione daret aequationem modularem quaesitam.
e integrationes a se invicem non pendent, ut facile tentanti ani-
Quamobrem ulterius progredi oportet et ultimam aequationem

denuo differentiare uti primam, ponendo quoque b v
autem hoc modo

$$dR = \frac{bdx\sqrt{(2x-xx)}}{\sqrt{[cc+bb(2x-xx)]}} + db\int\frac{ccdx\sqrt{(2x}}{[cc+bb(2x}$$

quod integrale iterum ita accipi debet, ut evanescat po

12. Ponatur iterum

$$S = \frac{dR}{db} - \frac{bdx\sqrt{(2x-xx)}}{db\sqrt{[cc+bb(2x-xx)]}};$$

erit

$$S = \int\frac{ccdx\sqrt{(2x-xx)}}{[cc+bb(2x-xx)]^{\frac{3}{2}}};$$

quae formula cum non sit integrabilis, videndum est, n
alterutra praecedentium vel ab utraque pendeat. Quod
$S+\alpha R+\beta z=Q$, ubi α et β ab x et z sint quanti
utcunque ex x et b et constantibus composita; debe
quantitas, ut evanescat posito $x=0$. Posito ergo b
$dQ=dS+\alpha dR+\beta dz$, ubi in differentiali ipsius Q b t
siderari debet.

13. At posito b constante est

$$dS = \frac{ccdx\sqrt{(2x-xx)}}{[cc+bb(2x-xx)]^{\frac{3}{2}}} \quad \text{et} \quad dR = \frac{bdx\sqrt{(}}{\sqrt{[cc+b}}$$

et

$$dz = \frac{dx\sqrt{[cc+bb(2x-xx)]}}{\sqrt{(2x-xx)}}.$$

Hanc ob rem erit

$$\frac{dQ}{dx} = \begin{bmatrix} cc(2x-xx)+\alpha bcc(2x-xx)+\alpha b^3(2 \\ +\beta c^4+2\beta b^2c^2(2x-xx)+\beta b^4(2x \end{bmatrix}$$

$$:[cc+bb(2x-xx)]^{\frac{3}{2}}\sqrt{(2x-xx)}$$

Ponatur ad similem formam obtinendam $Q = \frac{(\gamma x+\delta)\sqrt{(}}{\sqrt{[cc+bb(}}$
se evanescit posito $x=0$.

fferentietur nunc Q posito tantum x variabili; erit

$$[\gamma cc(2x-xx)+\gamma bb(2x-xx)^2+\gamma ccx+\delta cc-\gamma ccx^2-\delta ccx]$$
$$:[cc+bb(2x-xx)]^{\frac{3}{2}}\sqrt{(2x-xx)}.$$

denominatores iam sunt inter se aequales, fiant numeratores uales aequandis terminis, in quibus ipsius x similes sunt dimen-

I. $\gamma bb=\alpha b^3+\beta b^4$

II. $\gamma b^2=\alpha b^3+\beta b^4$

III. $4\gamma bb-2\gamma cc=4\alpha b^3+4\beta b^4-cc-\alpha bcc-2\beta b^2c^2$

IV. $3\gamma cc-\delta cc=2cc+2\alpha bcc+4\beta b^2c^2$

V. $\delta cc=\beta c^4$.

tur

$$\alpha=\frac{1}{b},\quad \beta=\frac{-1}{b^2+c^2},\quad \gamma=\frac{cc}{bb+cc}\quad \text{et}\quad \delta=\frac{-cc}{bb+cc}.$$

s ergo valoribus substitutis prodibit

$$\frac{cc(x-1)\sqrt{(2x-xx)}}{(bb+cc)\sqrt{[cc+bb(2x-xx)]}}=S+\frac{R}{b}-\frac{x}{b^2+c^2}$$

est

$$\frac{z}{b}-\frac{dx\sqrt{[cc+bb(2x-xx)]}}{db\sqrt{(2x-xx)}}\quad \text{et}\quad S=\frac{dR}{db}-\frac{bdx\sqrt{(2x-xx)}}{db\sqrt{[cc+bb(2x-xx)]}}$$

$b=a^2-c^2$ atque ideo $bb+cc=a^2$, $dx=\frac{ada-tda}{a^2}$ et $db=\frac{ada}{b}$,

$$Q=\frac{cc(t-a)\sqrt{(2at-tt)}}{a^3\sqrt{[a^2c^2+(a^2-c^2)(2at-tt)]}}$$

$$\frac{R}{b}=\frac{dz}{ada}-\frac{(adt-tda)\sqrt{[a^2c^2+(a^2-c^2)(2at-tt)]}}{a^3da\sqrt{(2at-tt)}}$$

atque

$$S = \frac{c^3 dz}{a^3 da} + \frac{a^2 - c^2}{a^3 da} d. \frac{dz}{da} - \frac{a^2 - c^2}{a^3 da} d. \frac{dt}{da} \sqrt{\frac{a^2c^2 \cdots}{}}$$

$$+ \frac{(2a^2 - 3c^2)(a\,dt - t\,da)}{a^5 da} \sqrt{\frac{a^2c^2 + (a^2 - cc)(2\cdots}{2at - tt}}$$

$$- \frac{(2aa - 2cc)(a\,dt - t\,da)}{a^3 da} \sqrt{\frac{2at - tt}{a^2c^2 + (a^2 - cc)(2\cdots}}$$

$$+ \frac{cc(a - t)(a^2 - c^2)(a\,dt - t\,da)^2}{a^3 da^2 (2at - tt)^{\frac{3}{2}} \sqrt{[a^2c^2 + (a^2 - cc)(2at - tt)]}}$$

16. Ne autem in nimis prolixos calculos incidar[...]
b, x et z; erit

$$S = \frac{1}{db} d. \frac{dz}{db} - \frac{1}{db} d. \frac{dx}{db} \sqrt{\frac{cc + bb(2x - xx)}{2x - xx}} - \frac{2b\,dx \cdots}{db}$$

$$+ \frac{cc\,dx^2(1 - x)}{db^2 (2x - xx)^{\frac{3}{2}} \sqrt{[cc + bb(2x - xx)]}}$$

His ergo loco S et R substitutis habebitur aequatio [...]

$$\frac{z}{bb + cc} = \frac{cc(1 - x)\sqrt{(2x - xx)}}{(bb + cc)\sqrt{[cc + bb(2x - xx)]}} - \frac{dx}{b\,db} \sqrt{\cdots}$$

$$- \frac{2b\,dx}{db} \sqrt{\frac{2x - xx}{cc + bb(2x - xx)}} + \frac{cc\,dx^2(1 \cdots}{db^2 (2x - xx)^{\frac{3}{2}} \sqrt{[cc \cdots}}$$

$$+ \frac{dz}{b\,db} + \frac{1}{db} d. \frac{dz}{db} - \frac{1}{db} d. \frac{dx}{db} \sqrt{\frac{cc + bb(2 \cdots}{2x - \cdots}}$$

Atque haec est aequatio differentialis secundi gradus, [...]
variabiles sunt positae. Ex hac autem aequatione[...]
solvuntur.

PROBLEMA 1

17. *Si curva EMN* (Fig. 2, p. 15) *ad axem AP* [...]
applicata quaeque PM aequalis sit quadranti AF [...]
coniugatorum alter sit ipsa abscissa AP, alter vero [...]
invenire aequationem inter abscissam AP et applicat[...]
curvae exprimentem.

cuum est curvam EMN transire per punctum E, quoniam eva-
emiaxe ellipsis AP quadrans ellipsis abit in alterum semiaxem
AE. Recta porro AT ad angulum
cum AP inclinata erit asymtotos
N, quia posito semiaxe AP infinite
drans ellipticus huic ipsi semiaxi fit
Ad aequationem autem inveniendam
c, $AP = t$ et $PM = AF = z$, atque
sa AP respectu ellipsis AF sit ae-
iaxi eius, erit haec quaestio casus
equationis inventae, quo est $l = a$
Posito ergo $x = 1$ abibit superior
hanc

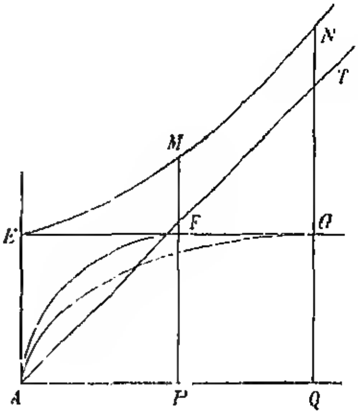


Fig. 2.

$$\frac{z}{bb+cc} = \frac{dz}{bdb} + \frac{1}{db} d.\frac{dz}{db}.$$

t $bb = a^2 - c^2 = t^2 - c^2$, erit $bdb = tdt$ et $db = \frac{tdt}{\sqrt{(tt-c^2)}}$. Atque
onstante erit $ddb = -\frac{ccdt^2}{(tt-cc)^{\frac{3}{2}}}$. Hinc ergo fit

$$d.\frac{dz}{db} = \frac{ddz\sqrt{(tt-cc)}}{tdt} + \frac{ccdz}{tt\sqrt{(tt-cc)}},$$

: haec aequatio

$$\frac{z}{tt} = \frac{dz}{tdt} + \frac{ddz(tt-cc)}{ttdt^2} + \frac{ccdz}{t^3dt}$$

$$tzdt^2 = (tt+cc)dtdz + tddz(tt-cc),$$

equatio quaesita pro curva proposita. Q. E. I.

equationem hanc sequenti modo ad differentialia primi gradus reduco
$= e^{\int sdt}$ existente $le = 1$; erit ergo

$$dz = e^{\int sdt}sdt \quad \text{et} \quad ddz = e^{\int sdt}(dsdt + ssdt^2).$$

oribus substituendis oritur sequens aequatio

$$tdt = (t^2 + c^2)sdt + t(tt-cc)ds + ts^2(t^2 - c^2)dt;$$

ac ita est comparata, ut nullis adhuc cognitis artificiis indeterminatae [illegible]
[illegible]cem separari possint. Interim vero constructio huius aequationis [illegible]
[illegible]ficationis ellipsis constat.

19. Ne vero cuiquam dubium oriatur, quod posito $t = 0$ fieri debeat [illegible]
[illegible] tamen superiores integrationes ita accipi debeant, ut posito $x = $ [illegible]
[illegible]que $z = 0$, monendum est, quod quidem in hoc casu, quo $z = c$, sit [illegible]
[illegible] vero est quoque $x = 0$, quia est $x = \frac{t}{a}$ et $t = a$ ideoque $x = 1$, [illegible]
hoc casu nusquam sit $x = 0$, propterea z uspiam evanescere debeat.

20. Quemadmodum in hoc problemate posuimus $t = a$, ita quoque [illegible]
[illegible]que aequatio inter t et a et constantes potest accipi et curva EMN [illegible]
[illegible], ut quaevis applicata PM aequalis sit respondenti arcui elliptico [illegible]
[illegible]bitur enim loco superioris aequationis haec aequatio

$$\frac{z}{a} = -\frac{(tt+ee)dz}{t^3 dt} + \frac{(tt-ee)ddz}{t\,dt^2} + T$$

[illegible]otante T eam ipsius t functionem, quae ex terminis aequationis ge[illegible]
[illegible] quibus non inest z, oritur, si loco x ponatur $\frac{t}{a}$ et loco b eius [illegible]
[illegible]$^2 - e^2)$ atque loco a eius valor in t ex aequatione inter a et t [illegible]
[illegible]stituatur. Neque vero haec aequatio tractatu est difficilior quam [illegible]
[illegible], in qua terminus T deest; reduci enim potest haec aequatio ut [illegible]
iam alibi[1]) ostendi.

PROBLEMA 2

21. *Datis infinitis ellipsibus AOF, ANG, AMH (Fig. 3, p. 17), quarum [illegible]*
[illegible]axis AE sit constans, alter vero variabilis ut AI, AK et AL, invenire [illegible]
[illegible]em pro curva $BONMC$, quae ab his omnibus ellipsibus arcus aequales [illegible]
[illegible], AM abscindat.

SOLUTIO

Ducta ad axem AC quacunque applicata MP curvae quaesitae [illegible]
[illegible] $= t$, $PM = u$ et $AE = e$; ellipsis vero AMH semiaxis variabilis [illegible]
[illegible] et arcus abscissus AM, qui est constantis quantitatis, sit $= f$.

1) Vide notam 3 p. 9. A. K.

$b = \sqrt{(a^2 - c^2)}$ erit $z = f$ et $u = c\sqrt{(2x - xx)}$. His igitur sub-
is aequatio inter z, x et b abit in hanc

$$\ldots cc \ldots \frac{cc(1-x)\sqrt{(2x-xx)}}{(bb+cc)\sqrt{[cc+bb(2x-xx)]}} - \frac{dx}{bdb}\sqrt{\frac{cc+bb(2x-xx)}{2x-xx}}$$

$$\ldots\frac{dx}{\ldots db}\sqrt{\frac{2x-xx}{cc+bb(2x-xx)}} + \frac{ccdx^2(1-x)}{db^2(2x-xx)^{\frac{3}{2}}\sqrt{[cc+bb(2x-xx)]}}$$

$$-\frac{1}{db}d.\frac{dx}{db}\sqrt{\frac{cc+bb(2x-xx)}{2x-xx}}.$$

$2x - xx = \frac{u^2}{c^2}$, multiplicetur ubique per

$$\sqrt{[cc+bb(2x-xx)]} = \frac{\sqrt{(c^4+bbuu)}}{c}$$

$$\ldots = \frac{cu(1-x)}{a^2} - \frac{c^3dx}{budb} - \frac{3budx}{cdb} + \frac{c^5dx^2(1-x)}{u^3db^2} - \frac{(c^4+bbuu)}{cudb}\,d.\frac{dx}{db}.$$

one si loco b substituatur $\frac{\sqrt{(tt-ccxx)}}{x}$ et propter $x = \frac{c-\sqrt{(cc-uu)}}{c}$
n aequatio differentialis secundi gradus inter t et u, nempe
rvae quaesitae. Q. E. I.

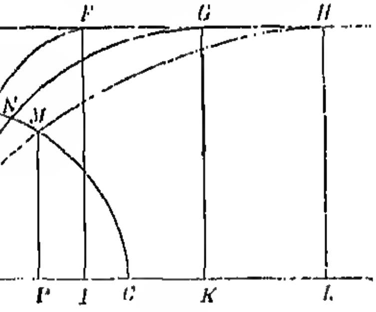


Fig. 3.

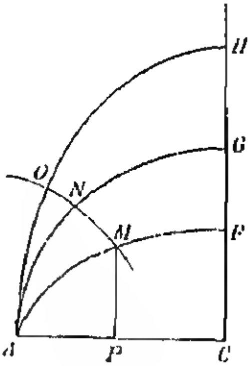


Fig. 4.

infinitae ellipses AMF, ANG et AOH (Fig. 4) omnes habeant
alem communem, ita ut C sit centrum omnium, pro hoc casu
quationem modularem erui oportet, antequam curvam MNO
quae ab omnibus arcus aequales AM, AN, AO abscindat. Sit

igitur $AC = c$, $CF = a$, $AP = t$, $PM = u$ et arcus $AM = z$

$$u = \frac{a}{c}\sqrt{(2ct - tt)} \quad \text{et} \quad du = \frac{acdt - atdt}{c\sqrt{(2ct - tt)}}$$

ideoque fiet

$$z = \int \frac{dt}{c}\sqrt{\frac{a^2c^2 + (cc - aa)(2ct - tt)}{2ct - tt}} = \int \frac{du}{a}\sqrt{\frac{a^4 - a^2u}{aa -}}$$

atque posito $u = ay$ erit

$$z = \int dy\sqrt{\left(a^2 + \frac{ccyy}{1 - yy}\right)},$$

quod integrale ita debet accipi, ut z evanescat posito $y = ($

23. Si haec denuo differentietur posito praeter y et a v

$$dz = dy\sqrt{\left(a^2 + \frac{ccyy}{1 - yy}\right)} + da\int \frac{ady}{\sqrt{\left(a^2 + \frac{ccyy}{1 - yy}\right)}}$$

atque posito

$$\frac{dz}{da} - \frac{dy}{da}\sqrt{\left(a^2 + \frac{ccyy}{1 - yy}\right)} = R$$

erit

$$R = \int \frac{ady}{\sqrt{\left(a^2 + \frac{ccyy}{1 - yy}\right)}}.$$

Hinc eodem modo fiet

$$dR = \frac{ady}{\sqrt{\left(a^2 + \frac{ccyy}{1 - yy}\right)}} + da\int \frac{ccyydy}{(1 - yy)\left(a^2 + \frac{ccyy}{1 - yy}\right)}$$

seu

$$\frac{dR}{da} - \frac{ady}{da\sqrt{\left(a^2 + \frac{ccyy}{1 - yy}\right)}} = \int \frac{ccyydy}{(1 - yy)\left(aa + \frac{ccyy}{1 - yy}\right)^{\frac{3}{2}}}$$

brevitatis gratia. Ponatur nunc $S + \alpha R + \beta z = Q$, ubi α e
tates ab y liberae, Q vero functio ipsarum a et y, quae
$y = 0$. Nunc ad α et β et Q invenienda differentietur hae
a constante; erit

$$\ldots yydy\sqrt{(1-yy)}\over \ldots(1-yy)+ccyy]^{\frac{3}{2}}} + \frac{\alpha ady\sqrt{(1-yy)}}{\sqrt{(a^2(1-yy)+ccyy)}} + \frac{\beta dy\sqrt{[a^2(1-yy)+ccyy]}}{\sqrt{(1-yy)}}$$

$$\left.\begin{matrix}\ldots ccyy - ccy^4 + \alpha a^3 - 2\alpha a^3y^2 + \alpha a^3y^4 + \alpha accyy - \alpha accy^4 \\ \ldots + \beta a^4 - 2\beta a^4y^2 + \beta a^4y^4 + 2\beta a^2c^2y^2 - 2\beta a^2c^2y^4 + \beta c^4y^4\end{matrix}\right) dy$$

$$:(a^2(1-yy)+ccyy)^{\frac{3}{2}}\sqrt{(1-yy)} = dQ.$$

t

$$Q = \frac{\gamma y\sqrt{(1-yy)}}{\sqrt{(a^2(1-yy)+ccyy)}}$$

huius differentiali posito a constanto et aequatis terminis homo-

$$\alpha a + \beta a^2 = \gamma, \quad \beta cc = -\gamma \quad \text{et} \quad 1 + \alpha a + 2\beta a^2 = 0.$$

$$\alpha = \frac{a^2+c^2}{a(a^2-cc)}, \quad \beta = \frac{-1}{a^2-c^2} \quad \text{et} \quad \gamma = \frac{cc}{aa-cc}.$$

loribus substitutis porvenietur tandem ad hanc aequationem

$$\frac{z}{\ldots-cc} = \frac{(a^2+c^2)dz}{ada(a^2-c^2)} + \frac{1}{da}d.\frac{dz}{da} - \frac{ccy\sqrt{(1-yy)}}{(a^2-c^2)\sqrt{(a^2(1-yy)+ccyy)}}$$

$$-\frac{(a^2+c^2)dy\sqrt{(a^2(1-yy)+ccyy)}}{a(a^2-c^2)da\sqrt{(1-yy)}} - \frac{2ady\sqrt{(1-yy)}}{da\sqrt{(a^2(1-yy)+ccyy)}}$$

$$\ldots\frac{ccydy^2}{da^2(1-yy)^{\frac{3}{2}}\sqrt{(a^2(1-yy)+ccyy)}} - \frac{1}{da}d.\frac{dy}{da}\sqrt{\frac{a^2(1-yy)+ccyy}{1-yy}},$$

eque sumtum est variabile ac y et z estque $y = \frac{u}{a}\cdot$

nunc ex infinitis ellipsibus, quarum omnium alter axis est con-
lter variabilis $2a$, construatur curva EMN (Fig. 2, p. 15) hac lege,
uo abscissae $AP = a$ respondeat applicata PM, quae aequalis est
lliptico sub semiaxibus a et c, hoc ergo casu erit $u = a$ et $y = 1$
$= z$. Quare posito da constanti habebitur pro curva EMN haec

$$azda^2 = (a^2 + c^2)dadz + a(aa - cc)ddz.$$

tio est ea ipsa, quam in solutione problematis 1 (§ 17) invenimus;
im hic casus cum illo problemate atque, quod ibi erat t, hic est a.

3*

26. *Descriptis infinitis ellipsibus AMF, ANG, AOH mune centrum C communemque verticem A habentibus invenire ab his omnibus ellipsibus arcus aequales AM, AN, AO absc*

SOLUTIO

Posito omnium harum ellipsium semiaxe constante cuiusvis AMF semiaxe altero variabili $CF = a$ atque cu $AP = t$ et applicata $PM = u$ fiat $\frac{u}{a} = y$ sitque longitu arcus AM, AN, AO aequales sumantur. His positis et collatis erit $z = f$ ideoque

$$\frac{f}{a^2 - e^2} + \frac{eey\sqrt{(1-yy)}}{(a^2-e^2)\sqrt{(a^2(1-yy)+eeyy)}} + \frac{(a^2+e^2)dy\sqrt{(a^2(}{a(a^2-e^2)da}$$

$$+ \frac{2ady\sqrt{(1-yy)}}{da\sqrt{(a^2(1-yy)+eeyy)}} + \frac{eeydy^2}{da^2(1-y^2)^{\frac{1}{2}}\sqrt{(a^2(1-yy}$$

$$+ \frac{1}{da}d.\frac{dy}{da}\sqrt{\frac{a^2(1-yy)+eeyy}{1-yy}} = 0$$

seu

$$\frac{f\sqrt{(1-yy)}}{(a^2-e^2)\sqrt{(a^2(1-yy)+eeyy)}} + \frac{eey(1-yy)}{(a^2-e^2)(a^2(1-yy)+eeyy)}$$

$$+ \frac{2ady(1-yy)}{da(a^2(1-yy)+eeyy)} + \frac{eeydy^2}{da^2(1-yy)(a^2(1-yy)+eeyy)} +$$

In qua aequatione si loco a ponatur $\frac{u}{y}$ et deinde loco y prodibit aequatio inter coordinatas t et u curvae quaesita

ANIMADVERSIONES IN RECTIFICATIONEM ELLIPSIS

Commentatio 154 indicis ENESTROEMIANI
Opuscula varii argumenti 2, 1750, p. 121—166

1. Ellipsis rectificatio tot iam variis methodis est frustra tentata, ut
um comparationem arcuum ellipticorum cum lineis rectis, sed etiam ne
cularibus quidem aut parabolicis expectare nequeamus. Cum enim for
differentialis, cuius integrale arcum ellipticum indefinitum exprimit,
do ab irrationalitate liberari queat, certum hoc est signum eius int
nem non solum non algebraice, sed etiam ne concessis quidem circ
erbolae quadraturis perfici posse. Quod cum tenendum sit de rectifica
psis indefinita, hinc adhuc minime sequitur arcum quempiam defi
uti totam perimetrum ellipsis omnem comparationem cum lineis vel
circularibus penitus respuere, propterea quod iam innumerabiles c
ignari possunt indefinite aeque parum rectificabiles atque ellipsis, in q
non arcus definiti per lineas rectas mensurari queant.

2. Missa igitur rectificatione ellipsis indefinita definitam potius
gressus, exploraturus, utrum tota cuiusque ellipsis perimeter non com
ssit ad mensuras cognitas, quorsum etiam logarithmos et arcus circ
ero, per expressiones finitas revocari. Quanquam autem in hac in
ione nihil admodum sum consecutus, quod scopo meo satisfecisset, t
eter expectationem nonnulla se mihi obtulerunt phaenomena satis singu
bus theoria linearum curvarum non mediocriter promoveri videtur.
o etiam difficultates, quae in toto hoc calculo occurrerunt, ansam
ebuerunt quaedam insignia artificia inveniendi, quae tam in calculo
li quam in theoria seriorum infinitarum ingentem utilitatem saepius a
sse videntur. Quamobrem operae pretium fore existimavi, si has spe
nes totumque quasi filum calculorum meorum dilucide exposuero.

3. *Super data recta AC* (Fig. 1) *tanquam altero sem…*
infinitos quadrantes ellipticos AP, AB, Ap, quorum ergo…

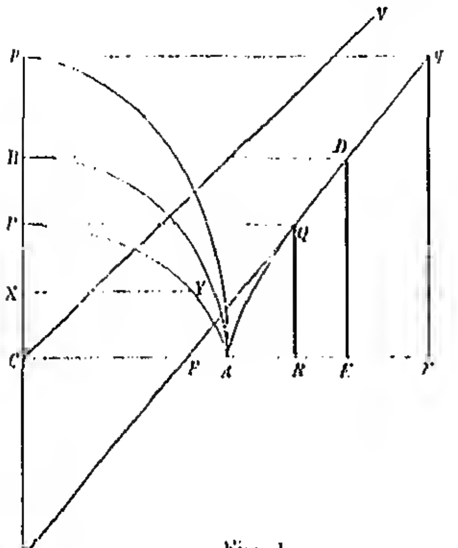


Fig. 1.

alteri vero semiaxes …
Cp. Tum ex singulis…
elliptici PA, BA, …
dantur, ita ut quad…
parallela et quadranti…
quod si ubique fieri …
Q sita erunt in linea…
cuius naturam invest…

Ad genesin hu…
attendenti mox pa…
habere proprietates,…
quam in ipsam h…
diligentius inquirar…
ductas saltem obite…

4. Primum igitur, si in recta indefinita CBp, qu…
normalis, capiatur quaevis abscissa CP, applicata PQ, …
aequalis quadranti perimetri ellipsis, cuius semiaxes …
data CA et ipsa abscissa CP. Hinc si capiatur abs…
casu quadrans ellipticus abibit in quadrantem circular…
spondens BD aequalis erit quartae parti peripheriae circ…
Unde, si ratio diametri ad peripheriam ponatur $= 1 :$ …
$BD = \frac{1}{2}\pi \cdot AC$, sive ob $\pi = 3,14159265358979 32$ erit

$$BD = 1,5707963267948966 . AC$$

5. Secundo: Si abscissa CP evanescat, ellipsis e…
…e cum linea recta confundetur. Hoc ergo casu qua…
ipsam lineam AC, cui propterea applicata abscissae …
aequalis. Quare ipsa recta CA erit applicata pu…

sita per punctum A transibit. Huius ergo curvae iam duo habe- cognita A et D, quorum alterum A geometrice datur, alterum rationem diametri ad peripheriam definitur.

tio: Ex cognito quovis curvae puncto Q intra A et D sito semper lam curvae punctum q ultra D situm definiri potest. Capiatur ertia proportionalis ad CP et CA, ut sit $Cp = \frac{CA \cdot CA}{CP}$; quia est $CA : Cp$, erit quadrans ellipticus Ap similis quadranti elliptico AP, ue eadem sit ratio inter semiaxes coniugatos. Hinc erit arcus Ap AP ut AC ad CP ideoque $pq : PQ = AC : CP$ seu $pq = \frac{AC \cdot PQ}{CP}$. er si curvae quaesitae arcus AD tantum iam fuerit descriptus, ex curvae pars Dq in infinitum extensa definietur.

irto: Hinc iam insignis proprietas aequationis, qua natura curvae rimitur, agnoscitur. Si enim recta data AC unitate designetur, $= 1$, abscissa autem quaevis unitate minor $CP = p$ eique re- pplicata $PQ = q$, tum vero ponatur abscissa illa altera $Cp = P$ et $q = Q$, erit $P = \frac{1}{p}$ et $Q = \frac{q}{p}$. Quare cum inter P et Q eadem aequatio, quae est inter p et q, patet aequationem inter p et q ationem esse subituram, si in ea loco p ubique scribatur $\frac{1}{p}$ et $\frac{q}{p}$ ido, qualis ipsius p functio sit q, coniicere licet.

nto: Patet crescentibus abscissis CP applicatas continuo crescere, cum maiores quam abscissae. Verum si abscissae statuantur infinitae, ipsis fient aequales; discrimen enim prodibit infinite parvum, unde quaesitam curvam habere asymtotam et quidem rectam CV angu- ACB bisecantem. Forma igitur huius curvae similis erit hy- equilaterae centrum in C, axem CA et asymtotam CV habentis. tione porro intelligitur curvam infra rectam CA productam sui re ideoque rectam CA eius fore diametrum perinde atque hy- Verumtamen hoc facile perspicitur nostram curvam multo lentius am suam CV appropinquare quam hyperbolam. Nam in hyperbola , cui nostram curvam comparamus, quaevis applicata PQ aequalis lineae AP; unde, cum applicata nostrae curvae arcui AP sit atet hyperbolam nostrae curvae fore circumscriptam, ita tamen, A et in spatio infinito se mutuo tangant.

9. His affectionibus latius patentibus in genere no
curvae naturam accuratius inquiramus ac proposita quacun
valorem respondentis applicatae $PQ = q$ investigemus;
finita contineri nequeat, per seriem infinitam exhiberi deb
resolvi oportet.

PROBLEMA

10. *Ex datis semiaxibus CA et CP quadrantis ellip*
infinitam definire longitudinem arcus quadrantis AYP.

SOLUTIO

Cum vocatus sit alter semiaxis $AC = 1$, alter ve
$AYP = q$, quaeratur primo arcus quivis indefinitus P
tum ducta ad CP applicata normali YX sit CX
ex natura ellipsis $x = p\sqrt{(1-yy)}$ hincque $dx = -\frac{p}{\sqrt{(1}}$
$ds = \sqrt{(dx^2+dy^2)}$

$$ds = \frac{dy\sqrt{(1-yy+ppyy)}}{\sqrt{(1-yy)}},$$

unde integrando erit arcus

$$s = \int\frac{dy\sqrt{(1-yy+ppyy)}}{\sqrt{(1-yy)}},$$

quae integratio ita institui debet, ut posito $y = 0$ fiat
evanescente applicata $XY = y$ simul $PY = s$ evanescit.
inventa si ponatur $y = CA = 1$, arcus indefinitus PY ab
quadrantis elliptici $PYA = q$, quem quaerimus, ita ut si

$$q = \int\frac{dy\sqrt{(1-yy+ppyy)}}{\sqrt{(1-yy)}},$$

tum peracta integratione ponatur $y = 1$.

[3] Totam ergo nostram non est necesse, ut
finiti, sed eum tantum, quem induit,
valor determinatus $= 1$; quo pacto

rem q exprimens obtineri poterit. Ponatur enim brevitatis gratia nn, ut sit $\sqrt{(1-yy+ppyy)} = \sqrt{(1-nnyy)}$, eritque hanc formulam evolvendo

$$\sqrt{(1-nnyy)} = 1 - \frac{1}{2}nnyy - \frac{1\cdot 1}{2\cdot 4}n^4y^4 - \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}n^6y^6 - \text{etc.}$$

valore substituto pro $\sqrt{(1-yy+ppyy)}$ arcus q ita exprimetur,

$$\frac{dy}{-yy)} - \frac{1}{2}nn\int\frac{yydy}{\sqrt{(1-yy)}} - \frac{1\cdot 1}{2\cdot 4}n^4\int\frac{y^4dy}{\sqrt{(1-yy)}} - \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}n^6\int\frac{y^6dy}{\sqrt{(1-yy)}}\text{ etc.,}$$

in singulis his integralibus post integrationem ponatur $y=1$.

Evolvamus ergo singula haec integralia; ac primo quidem ex circulo n est formulam

$$\int\frac{dy}{\sqrt{(1-yy)}}$$

arcum circuli, cuius sinus $=y$ pro radio $=1$; unde posito $y=1$ ula dabit quartam peripheriae partem, cuius radius $=1$. Ideoque ione diametri ad peripheriam $=1:\pi$ erit

$$\int\frac{dy}{\sqrt{(1-yy)}} = \frac{\pi}{2}$$

adepti sumus valorem primi termini in serie nostra ante inventa.

Reliqui termini pari modo per valorem π commode poterunt exprimi; enim termini integratio ad integrationem praecedentis reducitur; facilius intelligatur, consideremus formulam quamcunque $\int\frac{y^\mu dy}{\sqrt{(1-yy)}}$; ens $\int\frac{y^{\mu+2}dy}{\sqrt{(1-yy)}}$. Iam assumamus hanc formulam algebraicam $-yy)$; cuius differentiale cum sit

$$\frac{(\mu+1)y^\mu dy - (\mu+2)y^{\mu+2}dy}{\sqrt{(1-yy)}},$$

erit vicissim

$$(\mu+1)\int \frac{y^{\mu}dy}{\sqrt{(1-yy)}} - (\mu+2)\int \frac{y^{\mu+2}dy}{\sqrt{(1-yy)}} = y^{\mu+1}\sqrt{(1}$$

unde colligimus fore

$$\int \frac{y^{\mu+2}dy}{\sqrt{(1-yy)}} = \frac{\mu+1}{\mu+2}\int \frac{y^{\mu}dy}{\sqrt{(1-yy)}} - \frac{1}{\mu+2}y^{\mu+1}\sqrt{(1}$$

Quare invento integrali $\int \frac{y^{\mu}dy}{\sqrt{(1-yy)}}$ ex eo facile elicitur $\int \frac{y^{\mu+2}dy}{\sqrt{(1-yy)}}$.

14. Quoniam vero eos tantum horum integralium val
qui prodeunt posito $y = 1$, hoc casu quantitas algebraica

$$\frac{1}{\mu+2}y^{\mu+1}\sqrt{(1-yy)}$$

evanescit eritque generatim pro casu $y = 1$

$$\int \frac{y^{\mu+2}dy}{\sqrt{(1-yy)}} = \frac{\mu+1}{\mu+2}\int \frac{y^{\mu}dy}{\sqrt{(1-yy)}}.$$

Substituamus iam pro μ successive valores 0, 2, 4, 6, 8 etc
dimus esse

$$\int \frac{dy}{\sqrt{(1-yy)}} = \frac{\pi}{2},$$

erit, ut sequitur, si

$$\mu = 0,\quad \int \frac{y^{2}dy}{\sqrt{(1-yy)}} = \frac{1}{2}\int \frac{dy}{\sqrt{(1-yy)}} = \frac{1}{2}\cdot\frac{\pi}{2}$$

$$\mu = 2,\quad \int \frac{y^{4}dy}{\sqrt{(1-yy)}} = \frac{3}{4}\int \frac{y^{2}dy}{\sqrt{(1-yy)}} = \frac{1\cdot 3}{2\cdot 4}\cdot\frac{\pi}{2}$$

$$\mu = 4,\quad \int \frac{y^{6}dy}{\sqrt{(1-yy)}} = \frac{5}{6}\int \frac{y^{4}dy}{\sqrt{(1-yy)}} = \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot$$

$$\mu = 6,\quad \int \frac{y^{8}dy}{\sqrt{(1-yy)}} = \frac{7}{8}\int \frac{y^{6}dy}{\sqrt{(1-yy)}} = \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}$$

unde lex, qua sequentes progrediuntur, sponte elucet.

5. Quodsi iam isti valores pro formulis integralibus, ex quibus long
ntis elliptici q conflari inventa est, substituantur, reperietur

$$q = \frac{\pi}{2} - \frac{1}{2} nn \frac{1}{2} \cdot \frac{\pi}{2} - \frac{1 \cdot 1}{2 \cdot 4} n^4 \cdot \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{\pi}{2} - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} n^6 \cdot \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{\pi}{2} \quad \text{etc.},$$

ad sequentem seriem satis concinnam revocatur

$$q = \frac{\pi}{2}\left(1 - \frac{1 \cdot 1}{2 \cdot 2} n^2 - \frac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4} n^4 - \frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} n^6 - \text{etc.}\right),$$

lex progressionis est manifesta. Restituatur ergo pro nn suus
p eritque

$$\frac{\pi}{2}\left(1 - \frac{1 \cdot 1}{2 \cdot 2}(1 - pp) - \frac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4}(1 - pp)^2 - \frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6}(1 - pp)^3 - \text{e}\right.$$

6. Cum pro curva nostra $AQDq$ littera p exhibeat abscissam CP et
licatam PQ, iam adepti sumus pro ista curva aequationem inte
natas p et q, quae, etsi serie constat infinita, tamen non solum
m in se complectitur, sed etiam valores applicatae q mox sat
exhibet, si abscissa p parum ab unitate differat; hoc est, cu
$CA = 1$, si punctum P ipsi B fuerit proximum; tum eni
$p = nn$ quantitatem valde parvam series inventa valde convergit.

7. Hinc igitur indolem nostrae curvae prope punctum D, hoc es
ionem et curvaturam definire poterimus. Primo enim patet, ut
ns, si $p = 1$, fore $q = \frac{\pi}{2}$, ita ut sumta abscissa $CB = 1$ sit applic

$$BD = \frac{\pi}{2} = 1{,}5707963267948966.$$

o ad positionem tangentis inveniendam quaeratur ratio different
p, quae per differentiationem reperitur

$$\frac{dq}{dp} = \frac{\pi}{2} p \left\{ \begin{aligned} &\frac{1}{2} + \frac{1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4}(1 - pp) + \frac{1 \cdot 1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6}(1 - pp)^2 \\ &\qquad + \frac{1 \cdot 1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8}(1 - pp)^3 + \text{etc.} \end{aligned} \right\}$$

Posito iam $p = 1$ fiet $\frac{dq}{dp} = \frac{\pi}{4}$. Unde, si BG
puncto D, cum sit $BD : BG = dq : dp$, erit $BG = \frac{d}{d}$
$BD = \frac{\pi}{2}$ fiet $BG = 2 = 2BC$ et $CG = BC$. Sicque
BG erit dupla abscissae BC, et cum anguli BGD ta

$$= \frac{dq}{dp} = \frac{\pi}{4} = 0{,}78539816,$$

erit angulus $BGD = 38^{\circ}, 8', 45'', 41''', 51^{\mathrm{IV}}$.

18. Ad radium osculi seu evolutae in puncto D d
$\frac{dq}{dp} = \frac{\pi}{4}$ elementum curvae

$$\sqrt{(dp^2 + dq^2)} = dp \sqrt{\left(1 + \frac{\pi\pi}{16}\right)},$$

erit radius osculi

$$= \left(1 + \frac{\pi\pi}{16}\right)^{3:2} dp^3 : ddq.$$

At sumendis differentialibus secundis erit

$$\frac{ddq}{dp^3} = \frac{\pi}{2}\left(\frac{1}{2} + \frac{1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4}(1 - pp) + \frac{1 \cdot 1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6}(1 \cdot\right.$$

$$- \frac{\pi}{2} pp \left(\frac{1 \cdot 3}{2 \cdot 4} + \frac{1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 6}(1 - pp) + \frac{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 8}\right.$$

Posito ergo $p = 1$ erit

$$\frac{ddq}{dp^3} = \frac{\pi}{2}\left(\frac{1}{2} - \frac{3}{8}\right) = \frac{\pi}{16}.$$

Unde in puncto curvae D erit radius evolutae

$$= \frac{16}{\pi}\left(1 + \frac{\pi\pi}{16}\right)\sqrt{\left(1 + \frac{\pi\pi}{16}\right)},$$

meris proxime reperitur $= 10{,}470678.$[1]

[1]) 10,470672. Correxit A. K.

vero supra notavimus, si sit $P = \frac{1}{p}$, fore $Q = \frac{q}{p}$; quare his valoritutis impetrabimus novam aequationem inter p et q, qua natura iter exprimetur,

$$1 + \frac{1\cdot 1}{2\cdot 2}\frac{(1-pp)}{pp} - \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}\frac{(1-pp)^2}{p^4} + \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}\frac{(1-pp)^3}{p^6} - \text{etc.}\Big);$$

m ante inventa combinetur, innumerabiles aliae novae aequationes oterunt. Veluti si prior per p multiplicata ab hac subtrahatur,

$$q = \frac{\pi}{2}p\Big(\frac{1\cdot 1}{2\cdot 2}\frac{(1-pp)(1+pp)}{pp} - \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}\frac{(1-pp)^2(1-p^4)}{p^4} + \text{etc.}\Big),$$

itur ad hanc

$$\Big(\frac{1}{2}\cdot\frac{1+pp}{p} - \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 4}\frac{(1-p^4)(1-pp)}{p^3} + \frac{1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 4\cdot 4\cdot 6\cdot 6}\frac{(1+p^6)(1-pp)^2}{p^5} - \text{etc.}\Big).$$

eries adhuc sit divisibilis per $\frac{1+pp}{2p}$, erit

$$p)\frac{(1+pp)}{p}\left\{\begin{array}{l}1 - \frac{1\cdot 3}{4\cdot 4}\frac{(1-pp)}{pp}(1-pp) + \frac{1\cdot 3\cdot 3\cdot 5}{4\cdot 4\cdot 6\cdot 6}\frac{(1-pp+p^4)}{p^4}(1-pp)^2\\ \qquad - \frac{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7}{4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8}\frac{(1-pp+p^4-p^6)}{p^6}(1-pp)^3 + \text{etc.}\end{array}\right\}$$

anifestum autem est has series parum subsidii afferre, si applicatas elimus, quae longius a BD, quae abscissae $p=1$ respondet, sint i enim pro p ponatur numerus vel valde magnus vel valde parvus, nta vel parum admodum convergit vel etiam divergit. Si enim udinem primae applicatae CA, quae abcissae $p=0$ respondet, demus, serie primum inventa uti conveniet, quia in reliquis termini finito magni. Habebimus igitur pro hoc casu $p=0$

$$\Big(1 - \frac{1\cdot 1}{2\cdot 2} - \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4} - \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6} - \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8} - \text{etc.}\Big),$$

quae tam lente convergit, ut, etiamsi plurimi termini a
verus ipsius q valor, quem novimus esse $= 1$, inde d

21. Quanquam autem nunc quidem novimus ess

$$1 - \frac{1\cdot 1}{2\cdot 2} - \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4} - \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6} - \text{e}$$

tamen inventio summae huius seriei non parum ar
tentetur. Veritatem quidem ex formula, quam quo
culi quadratura dedit[1]), intelligere licet, si termini
gantur; sic enim prodit

$$1 - \frac{1\cdot 1}{2\cdot 2} = \frac{1\cdot 3}{2\cdot 2},$$

$$\frac{1\cdot 3}{2\cdot 2} - \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4} = \frac{1\cdot 3\cdot (4\cdot 4 - 1\cdot 1)}{2\cdot 2\cdot 4\cdot 4} = \frac{1}{2}$$

$$\frac{1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4} - \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6} = \frac{1\cdot 3\cdot 3\cdot 5\cdot}{2\cdot 2\cdot 4\cdot 4}$$

unde valor seriei in infinitum continuatae erit

$$\frac{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdot 9\cdot 11\cdot 11\cdot 13\cdot 13}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdot 10\cdot 10\cdot 12\cdot 12\cdot 14}$$

quae expressio cum sit ipsa Wallisiana, patet sum
$= \frac{2}{\pi}$. Interim tamen iuvabit tradere methodum han
a priori summandi.

PROBLEMA

22. *Invenire summam huius seriei infinitae*

$$1 - \frac{1\cdot 1}{2\cdot 2} - \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4} - \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}$$

gressionis primo intuitu est manifesta.

llis (1616—1703), *Arithmetica infinitorum sive no*
draturam aliaque difficiliora Matheseos problemata;
K.

...atineantur. Cuiusmodi est haec

$$\frac{1}{\sqrt{(1-xx)}} = 1 + \frac{1}{2}xx + \frac{1\cdot3}{2\cdot4}x^4 + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}x^6 + \frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}x^8 + \text{etc.}$$

...per differentiale quodpiam dP multiplicando et integrando

$$\int\frac{dP}{\sqrt{(1-xx)}} = P + \frac{1}{2}\int xx\,dP + \frac{1\cdot3}{2\cdot4}\int x^4 dP + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\int x^6 dP + \text{etc.}$$

...rentiale hoc dP ita definiatur, ut, si post integrationem ponatur

$$\int xx\,dP = \frac{1}{2}P$$

$$\int x^4 dP = \frac{1}{4}\int xx\,dP = \frac{1\cdot1}{2\cdot4}P$$

$$\int x^6 dP = \frac{3}{6}\int x^4 dP = \frac{1\cdot1\cdot3}{2\cdot4\cdot6}P$$

$$\int x^8 dP = \frac{5}{8}\int x^6 dP = \frac{1\cdot1\cdot3\cdot5}{2\cdot4\cdot6\cdot8}P;$$

...ui hi valores substituantur, habebitur

$$\int\frac{dP}{\sqrt{(1-xx)}} = P\left(1 + \frac{1\cdot1}{2\cdot2} + \frac{1\cdot1\cdot1\cdot3}{2\cdot2\cdot4\cdot4} + \frac{1\cdot1\cdot1\cdot3\cdot3\cdot5}{2\cdot2\cdot4\cdot4\cdot6\cdot6} + \text{etc.}\right)$$

$$\int\frac{dP}{\sqrt{(1-xx)}} = Ps,$$

... post integrationem statuatur $x = 1$.

...Huc ergo res redit, ut quaeratur formula differentialis dP, ...
...us conditionibus satisfiat seu ut in genere sit

$$\int x^{\mu+2} dP = \frac{\mu-1}{\mu+2}\int x^{\mu} dP,$$

si quidem post integrationem utramque ponatur x
conditione sit

$$\int x^{\mu+2} dP = \frac{\mu-1}{\mu+2} \int x^{\mu} dP + \frac{Qx^{\mu+}}{\mu+}$$

ubi Q eiusmodi sit functio ipsius x, quae evanescat
ergo differentialia eritque per x^{μ} dividendo

$$xxdP = \frac{\mu-1}{\mu+2} dP + \frac{xdQ+(\mu+1)}{\mu+2}$$

seu

$$0 = (\mu-1)dP - (\mu+2)xxdP + xdQ +$$

quae aequatio, cum locum habere debeat pro omn
vetur in has duas

$$0 = dP - xxdP + Qdx$$

$$0 = -dP - 2xxdP + xdQ +$$

unde fit

$$dP = \frac{-Qdx}{1-xx} = \frac{xdQ+Qdx}{1+2xx}$$

et

$$xdQ(1-xx) = -Qdx(2+x$$

Quare cum sit

$$\frac{dQ}{Q} = -\frac{dx(2+xx)}{x(1-xx)} = -\frac{2dx}{x} - \frac{3}{1}$$

erit

$$Q = -\frac{(1-xx)^{\frac{3}{2}}}{xx} \quad \text{et} \quad dP = \frac{dx}{xx}\sqrt{(1}$$

24. Verum hic notandum est, etsi valor ipsius
tamen casu $\mu = 0$ quantitatem algebraicam $\frac{Qx^{\mu+1}}{\mu+2}$ n
$x = 0$; quae tamen conditio aeque est necessaria at
non sit $\int xxdP = -\frac{1}{2}P$. Cum autem reliquae form
habeant, a formula $\int xxdP$ erit incipiendum eritqu

$$\int x^4 dP = \frac{1}{4}\int xxdP$$

$$\int x^6 dP = \frac{3}{6}\int x^4 dP = \frac{1\cdot 3}{4\cdot 6}\int xxd$$

$$\int x^8 dP = \frac{5}{8}\int x^6 dP = \frac{1\cdot 3\cdot 5}{4\cdot 6\cdot 8}\int x$$

etc.,

...itur

$$\int\frac{dP}{\sqrt{(1-xx)}}=P+\int xxdP\left(\frac{1}{2}+\frac{1\cdot1\cdot3}{2\cdot4\cdot4}+\frac{1\cdot1\cdot3\cdot3\cdot5}{2\cdot4\cdot4\cdot6\cdot6}+\text{etc.}\right).$$

$$\frac{1}{2}+\frac{1\cdot1\cdot3}{2\cdot4\cdot4}+\frac{1\cdot1\cdot3\cdot3\cdot5}{2\cdot4\cdot4\cdot6\cdot6}+\text{etc.}=2(1-s)$$

$$\int\frac{dP}{\sqrt{(1-xx)}}=P+2(1-s)\int xxdP.$$

$=\frac{dx}{xx}\sqrt{(1-xx)}$ erit

$$P=C-\frac{\sqrt{(1-xx)}}{x}-\text{A}\sin x,$$

$$\int xxdP=\int dx\sqrt{(1-xx)}=\frac{1}{2}\text{A}\sin x+\frac{1}{2}x\sqrt{(1-xx)}$$

$$\int\frac{dP}{\sqrt{(1-xx)}}=D-\frac{1}{x},$$

...tes C et D ita accipi debent, ut integralia haec evanescant posito ...nquam autem utraque seorsim fit infinita, tamen coniunctae se ...ruent. Erit enim

$$\int\frac{dP}{\sqrt{(1-xx)}}-P=D-\frac{1}{x}-C+\frac{\sqrt{(1-xx)}}{x}+\text{A}\sin x;$$

...nescat posito $x=0$, debet esse $D=C$ ideoque posito iam $x=1$ fiet

$$\int\frac{dP}{\sqrt{(1-xx)}}-P=-1+\frac{\pi}{2},$$

...dem hoc casu est $\int xxdP=\frac{\pi}{4}$, prodibit

$$-1+\frac{\pi}{2}=2(1-s)\frac{\pi}{4}=\frac{\pi}{2}-\frac{\pi}{2}s$$

...lligitur fore $\frac{\pi}{2}s=1$ et $s=\frac{2}{\pi}$ seu

$$1-\frac{1\cdot1}{2\cdot2}-\frac{1\cdot1\cdot1\cdot3}{2\cdot2\cdot4\cdot4}-\frac{1\cdot1\cdot1\cdot3\cdot3\cdot5}{2\cdot2\cdot4\cdot4\cdot6\cdot6}-\text{etc.}=\frac{2}{\pi},$$

...natura iam conclusimus.

25. Quoniam igitur eruimus in ipso initio esse
indolem huius curvae prope punctum A indagem
catae q inquiramus, si abscissa p fuerit valde par
mus iterum $1-pp=nn$, et cum sit

$$q=\frac{\pi}{2}\left(1-\frac{1\cdot1}{2\cdot2}nn-\frac{1\cdot1\cdot1\cdot3}{2\cdot2\cdot4\cdot4}n^4-\frac{1\cdot1\cdot1\cdot}{2\cdot2\cdot4\cdot}\right.$$

et quia novimus fore proxime $q=1$, addamus aeq

$$0=1-\frac{\pi}{2}\left(1-\frac{1\cdot1}{2\cdot2}-\frac{1\cdot1\cdot1\cdot3}{2\cdot2\cdot4\cdot4}-\frac{1\cdot1\cdot1\cdot}{2\cdot2\cdot4\cdot}\right.$$

atque habebimus

$$q=1+\frac{\pi}{2}\left(\frac{1\cdot1}{2\cdot2}(1-nn)+\frac{1\cdot1\cdot1\cdot3}{2\cdot2\cdot4\cdot4}(1-n^4)+\frac{1\cdot1\cdot}{2\cdot2\cdot}\right.$$

cuius seriei cum singuli termini sint per $1-nn=$
haec expressio ad hanc

$$q=1+\frac{\pi}{8}pp\left\{\begin{aligned}&1+\frac{1\cdot3}{4\cdot4}(1+nn)+\frac{1\cdot3\cdot3\cdot5}{4\cdot4\cdot6\cdot6}\\&+\frac{1\cdot3\cdot3\cdot5\cdot5\cdot7}{4\cdot4\cdot6\cdot6\cdot8\cdot8}(1+n^2+\end{aligned}\right.$$

26. Quodsi in hac expressione singuli termin
evolvantur, reperietur

$$q=1+\frac{\pi}{2}pp\left\{\begin{aligned}&+\frac{1\cdot1}{2\cdot2}+\frac{1\cdot1\cdot1\cdot3}{2\cdot2\cdot4\cdot4}+\frac{1\cdot1\cdot1\cdot3\cdot3\cdot5}{2\cdot2\cdot4\cdot4\cdot6\cdot6}+\\&+n^2\left(\frac{1\cdot1\cdot1\cdot3}{2\cdot2\cdot4\cdot4}+\frac{1\cdot1\cdot1\cdot3\cdot3\cdot5}{2\cdot2\cdot4\cdot4\cdot6\cdot6}+\frac{1\cdot1}{2\cdot2}\right.\\&+n^4\left(\frac{1\cdot1\cdot1\cdot3\cdot3\cdot5}{2\cdot2\cdot4\cdot4\cdot6\cdot6}+\frac{1\cdot1\cdot1\cdot3\cdot3\cdot5\cdot5}{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdot8}\right.\\&+n^6\left(\frac{1\cdot1\cdot1\cdot3\cdot3\cdot5\cdot5\cdot7}{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdot8\cdot8}+\text{etc.}\right)\\&\qquad\qquad\qquad\text{etc.}\end{aligned}\right.$$

At ex supra inventis habemus summam prima

$$\frac{1\cdot1}{2\cdot2}+\frac{1\cdot1\cdot1\cdot3}{2\cdot2\cdot4\cdot4}+\frac{1\cdot1\cdot1\cdot3\cdot3\cdot5}{2\cdot2\cdot4\cdot4\cdot6\cdot6}+\text{etc.}$$

$$\frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6} + \text{etc.} = \frac{1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4} - \frac{2}{\pi}$$

o coefficiens ipsius n^6 erit

$$= \frac{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6} - \frac{2}{\pi}$$

icque tandem obtinebitur

$$+ \frac{\pi}{2}pp\left\{\begin{array}{c}\left(1-\frac{2}{\pi}\right) + \left(\frac{1\cdot 3}{2\cdot 2} - \frac{2}{\pi}\right)nn + \left(\frac{1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4} - \frac{2}{\pi}\right)n^4 \\ + \left(\frac{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6} - \frac{2}{\pi}\right)n^6 + \text{etc.}\end{array}\right\}$$

$$+ pp\left\{\begin{array}{c}\left(\frac{\pi}{2}-1\right) + \left(\frac{1\cdot 3}{2\cdot 2}\cdot\frac{\pi}{2} - 1\right)nn + \left(\frac{1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4}\cdot\frac{\pi}{2} - 1\right)n^4 \\ + \left(\frac{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}\cdot\frac{\pi}{2} - 1\right)n^6 + \text{etc.}\end{array}\right\}$$

mus iam hic $n = 1$, ut obtineamus aequationem huius formae qua natura curvae prope punctum A exprimitur; cum enim at verum aequationem futuram esse huius formae

$$q = 1 + App + Bp^4 + Cp^6 + Dp^8 + \text{etc.},$$

valde parva assumatur, reliqui termini praeter binos primos unt atque ex aequatione $q = 1 + App$ tam positio tangentis ra in puncto A colligi poterit. Posito enim $AR = x$, $RQ = y$; et $p = y$ ideoque, si arcus AQ fuerit minimus, is cum paratur, cuius aequatio $x = Ayy$ seu $yy = \frac{1}{A}x$ ac propterea $\frac{1}{A}$ para- sequitur tangentem curvae in A fore ad rectam AC perpendidium osculi ibidem esse $= \frac{1}{2A}$.

28. Hic igitur coefficiens A reperietur, si i
quantitas pp multiplicatur, ponatur $n = 1$, ita m

$$A = \left(\frac{\pi}{2} - 1\right) + \left(\frac{1 \cdot 3}{2 \cdot 2} \cdot \frac{\pi}{2} - 1\right) + \left(\frac{1 \cdot 3 \cdot 3}{2 \cdot 2 \cdot 4}\right.$$

quae autem, si eius summatio tentetur, tam par
ut eius summam adeo infinitam suspicari debean
eo magis confirmamur, si seriem primo (§ 15) inv
ipsius p evolvamus, unde fit

$$q = \frac{\pi}{2}\begin{cases} 1 - \frac{1 \cdot 1}{2 \cdot 2} - \frac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4} - \frac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4} \\ + pp\left(\frac{1 \cdot 1}{2 \cdot 2} + \frac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4} \cdot 2 + \frac{1 \cdot 1}{2 \cdot 2}\right. \\ - p^4\left(\frac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4} + \frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6}\right. \\ \text{etc.} \end{cases}$$

29. Hinc ergo coefficiens ipsius pp in aequ

$$q = 1 + App + Bp^4 + Cp^6 +$$

erit

$$A = \frac{\pi}{2}\left(\frac{1 \cdot 1}{2 \cdot 2} \cdot 1 + \frac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4} \cdot 2 + \frac{1 \cdot 1 \cdot 1}{2 \cdot 2 \cdot 4}\right.$$

seu

$$A = \frac{\pi}{4}\left(\frac{1}{2} + \frac{1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4} + \frac{1 \cdot 1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6} + \frac{1 \cdot 1}{2 \cdot 2}\right.$$

similique modo et reliquos coefficientes B, C,
licebit. Verum hoc labore supersedere poterim
coefficientem A, sed etiam omnes reliquos pr
spicuum hoc fiet ex solutione huius problematis

PROBLEMA

30. *Invenire summam huius seriei infinitae*

$$s = \frac{1}{2} + \frac{1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4} + \frac{1 \cdot 1 \cdot 3 \cdot 3 \cdot}{2 \cdot 2 \cdot 4 \cdot 4 \cdot}$$

SOLUTIO

...atur ad hanc summam s inveniendam haec formula

$$\frac{1}{\sqrt{(1-xx)}} = 1 + \frac{1}{2}xx + \frac{1\cdot 3}{2\cdot 4}x^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \text{etc.},$$

$$\frac{dP}{\ldots 1-xx)} = P + \frac{1}{2}\int xxdP + \frac{1\cdot 3}{2\cdot 4}\int x^4dP + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\int x^6dP + \text{etc.},$$

...post integrationes singulas ponatur $x = 1$.

$$\int xxdP = \frac{3}{4}P$$

$$\int x^4\,dP = \frac{5}{6}\int xxdP = \frac{3\cdot 5}{4\cdot 6}P$$

$$\int x^6\,dP = \frac{7}{8}\int x^4\,dP = \frac{3\cdot 5\cdot 7}{4\cdot 6\cdot 8}P$$

etc.

et

$$\int\frac{dP}{\sqrt{(1-xx)}} = P\left(1 + \frac{1\cdot 3}{2\cdot 4} + \frac{1\cdot 3\cdot 3\cdot 5}{2\cdot 4\cdot 4\cdot 6} + \frac{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7}{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8} + \text{etc.}\right)$$

$$\int\frac{dP}{\sqrt{(1-xx)}} = 2Ps;$$

...nto P reperietur s, si post integrationem ponatur $x = 1$.

Cum igitur generaliter esse debeat

$$\int x^{\mu+2}dP = \frac{\mu+3}{\mu+4}\int x^\mu dP + \frac{x^{\mu+1}Q}{\mu+4},$$

... Q eiusmodi sit functio, quae evanescat posito $x = 1$, erit

$$(\mu+4)xxdP = (\mu+3)dP + xdQ + (\mu+1)Qdx,$$

...c sequentes aequationes conficiuntur

$$xxdP = dP + Qdx$$

$$4xxdP = 3dP + xdQ + Qdx$$

$$dP = \frac{-Qdx}{1-xx} = \frac{-xdQ - Qdx}{3-4xx}$$

hincque elicitur

$$\frac{dQ}{Q} = \frac{2dx - 3xxdx}{x(1-xx)} = \frac{2\,dx}{x} - \frac{xdx}{1-x}$$

et

$$Q = -xx\sqrt{(1-xx)}.$$

Quare habebitur

$$dP = \frac{xxdx}{\sqrt{(1-xx)}} \quad \text{et} \quad \frac{dP}{\sqrt{(1-xx)}} = \frac{xxdx}{1-xx} = -$$

Fiet ergo $P = \frac{1}{4}\pi$, si post integrationem ponatur x

$$\int \frac{dP}{\sqrt{(1-xx)}} = -x + \frac{1}{2}\, l\, \frac{1+x}{1-x}$$

cuius valor posito $x = 1$ fit utique infinitus. Erit ig
seriei propositae infinite magna.

32. Quia igitur coefficiens A ipsius pp in aequa

$$q = 1 + Ap^2 + Bp^4 + Cp^6 + \text{et}$$

est infinitus, radius osculi curvae in puncto A uti
Verum praeterea haec aequatio, in qua omnes omni
D etc. fiunt infiniti, nihil plane ad curvae cognitio
radius osculi curvae in A est infinite parvus, natura
huiusmodi aequatione $q = 1 + ap^m$ exprimetur, in
sit minor, verumtamen unitate maior; sed ex omn
tradita, nulla via patet, qua hunc exponentem m
enim is numerus integer esse nequeat, nulla seriern
ita est comparata, ut ex ea potestatem ipsius p i

intelligimus problema esse summoper
aris requiritur, quae naturam curvae
unctum A exhibeat. Notum est enir
nque fuerit curva AQ, naturam minim
smodi aequatione $y^m = Ax$ comprehen
pro curvis autem transcendentibus
portiunculas cum arcubus curvarum

in nostra curva, etsi est transcendens, hoc eo magis mirum quod nulla huiusmodi formula $y^m = Ax$ exhiberi possit, quae e eius portiunculae circa A sitae naturam declaret.

nodum ut resolvamus, aequationem nobis finitam inter coordi- investigare oportebit, quae etsi, ut facile praevidere licet, ad ecundi ordinis exsurget, tamen ad accuratiorem curvae cogni- erit accommodata. Eliciemus autem huiusmodi aequationem, erminorum finito constet, si seriem primo inventam ad summam Cum enim posito $1 - pp = nn$ sit

$$= 1 - \frac{1\cdot 1}{2\cdot 2}nn - \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}n^4 - \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}n^6 - \text{etc.},$$

ndo

$$\frac{dq}{dn} = -\frac{1\cdot 1}{2}n - \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4}n^3 - \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6}n^5 - \text{etc.},$$

ultiplicata denuoque differentiata dat

$$d.\frac{ndq}{dn} = -1\cdot 1n - \frac{1\cdot 1}{2\cdot 2}\cdot 1\cdot 3n^3 - \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}\cdot 3\cdot 5n^5 - \text{etc.}$$

aec per $\frac{dn}{n}$ ac rursus integretur; erit

$$\int\frac{1}{n}\,d.\frac{ndq}{dn} = -1n - \frac{1\cdot 1}{2\cdot 2}\cdot 1n^3 - \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}\cdot 3n^5 - \text{etc.}$$

per $\frac{dn}{n^3}$ et integrando prodibit

$$\frac{2}{\pi}\int\frac{dn}{n^3}\int\frac{1}{n}\,d.\frac{ndq}{dn} = \frac{1}{n} - \frac{1\cdot 1}{2\cdot 2}n - \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}n^3 - \text{etc.},$$

un sit ipsa proposita per n divisa, erit

$$\int\frac{dn}{n^3}\int\frac{1}{n}\,d.\frac{ndq}{dn} = \frac{2q}{\pi n} \quad \text{seu} \quad \int\frac{dn}{n^3}\int\frac{1}{n}\,d.\frac{ndq}{dn} = \frac{q}{n}.$$

umus nunc differentialia habebiturque

$$\int\frac{1}{n}\,d.\frac{ndq}{dn} = \frac{ndq - qdn}{nn} \quad \text{seu} \quad \int\frac{1}{n}\,d.\frac{ndq}{dn} = \frac{nndq}{dn} - nq$$

porroque differentiando

$$\frac{1}{n} d. \frac{ndq}{dn} = nd. \frac{ndq}{dn} + ndq - ndq - q$$

seu

$$(1 - nn) d. \frac{ndq}{dn} + qndn = 0.$$

Iam ob $1 - nn = pp$ erit

$$ndn = -pdp \quad \text{et} \quad \frac{dn}{n} = -\frac{pdp}{1 - pp},$$

unde fit

$$-ppd. \frac{(1 - pp)dq}{pdp} - pqdp = 0 \quad \text{seu} \quad d. \frac{(1 - pp)d}{pdp}$$

Sumatur iam dp constans; erit

$$\frac{(1 - pp)ddq}{pdp} - \frac{dq(1 + pp)}{pp} + \frac{qdp}{p} = 0$$

seu

$$p(1 - pp)ddq - dpdq(1 + pp) + pqdp^2$$

36. En igitur aequationem differentialem secundi g
posita

$$p(1 - pp)ddq - dpdq(1 + pp) + pqdp^2$$

ex qua potestas illa ipsius p in aequatione $q = 1 + A$
scissa p valde parva statuatur. Cum igitur fiat

$$dq = mAp^{m-1}dp \quad \text{et} \quad ddq = m(m - 1)A$$

orietur

$$\begin{aligned} &m(m - 1)Ap^{m-1} - \quad mAp^{m-1} + p \\ &- m(m - 1)Ap^{m+1} - \quad mAp^{m+1} + Ap^{m+1} \end{aligned}$$

seu

$$m(m - 2)Ap^{m-1} - (mm - 1)Ap^{m+1} + p$$

rgo esse $m = 2$, ut terminus Ap^{m-1} cum p
obtinetur $A = \infty$; praeterea vero hinc pers
numerum fractum esse posse, ita ut hinc
ri potius quam tolli videatur.

Quodsi regulis consuetis uti velimus ad aequationem inventam in
olvendam, quae secundum potestates ipsius p procedat, quoniam
rimum seriei terminum esse $=1$, nullam aliam formam inde col-
 nisi hanc

$$q = 1 + Ap^2 + Bp^4 + Cp^6 + Dp^8 + \text{etc.},$$

$$\frac{dq}{dp} = 2Ap + 4Bp^3 + 6Cp^5 + 8Dp^7 + \text{etc.}$$

$$\frac{ddq}{dp^2} = 2A + 12Bp^2 + 30Cp^4 + 56Dp^6 + \text{etc.},$$

s in aequatione substituti praebebunt

$$\left.\begin{array}{rrrrrr} +\,2Ap & +\,12Bp^3 & +\,30Cp^5 & +\,56Dp^7 & + & \text{etc.} \\ & -\,2A & -\,12B & -\,30C & - & \text{etc.} \\ -\,2A & -\,4B & -\,6C & -\,8D & - & \text{etc.} \\ & -\,2A & -\,4B & -\,6C & - & \text{etc.} \\ +\,1 & +\,A & +\,B & +\,C & + & \text{etc.} \end{array}\right\} = 0,$$

s coefficientes A, B, C etc. prodeunt infiniti.

inc igitur videmus regulas ordinarias, secundum quas vulgo forma
quam aequatio differentialis transmutanda sit, diiudicari solet, non
ientes, cum hoc casu nullam afferant utilitatem; unde nostra
o maiorem meretur attentionem. Sequenti tamen modo ex ea
vae prope punctum A colligi poterit, ex quo simul intelligetur,
dum quoque in aliis casibus defectus isto regularum usu recep-
pleri eaeque ad praxin accommodari debeant. Quia enim abscissam
infinito parva habemus, in aequatione pro $1-pp$ et $1+pp$ ponere
t quia novimus esse hoc casu proximo $q=1$, pro quantitate finita
scribamus; quo facto aequatio differentio-differentialis inventa pro
abscissa p est minima, sequentem induet formam

$$pddq - dpdq + pdp^2 = 0.$$

39. Huius iam aequationis resolutio est facilis;
stans, ponatur $dq = rdp$; erit $ddq = drdp$ habebiturq

$$pdr - rdp + pdp = 0$$

sive

$$\frac{pdr - rdp}{pp} + \frac{dp}{p} = 0,$$

cuius integrale est $\frac{r}{p} + lp = C$, unde fit $r = Cp - $

$$dq = Cpdp - pdplp.$$

Haec iam aequatio integrata dabit

$$q = 1 + \frac{1}{2}Cp^2 - \frac{1}{2}pplp + \frac{1}{4}$$

in qua cum terminus pp incomparabiliter sit mino
curvae initio A

$$q = 1 - \frac{1}{2}pplp.$$

40. Nunc igitur naturam curvae prope initium A
nire possumus; si enim vocemus $AR = x$ et $RQ = y$
orietur haec $x = -\frac{1}{2}yyly$, ad quam aequatio gener
si coordinatae x et y sint quam minimae. Patet igi
arculum circa A tanquam portiunculam curvae al
sed eius naturam logarithmos implicare. Et quoni
in exponentialem transformari potest, initium curv
erit cum linea transcendente, cuius aequatio est
numero, cuius logarithmus hyperbolicus est $= 1$.

1. Aequatione hac $x = -\frac{1}{2}yyly$ confirmantur q
ibus huius curvae in puncto A notavimu
fore quoque $yyly$ ac proinde $x = 0$, ets
sit $dx = -ydyly - \frac{1}{2}ydy$, quia y in
it $dx = -ydyly$ ac propterea $\frac{dy}{dx} =$
n curvae in A ad abscissam AR ess

bnormalis $\frac{y\,dy}{dx} = \frac{-1}{ly}$ hocque casu subnormalis radio evolutae
o $ly = \infty$, si $y = 0$, manifestum est radium osculi curvae in A
parvum.

xime autem differt haec curva a curvis algebraicis, quae in initio A
nt radium osculi evanescentem. Curvarum enim algebraicarum,
dole gaudent, natura circa initium A huiusmodi formula expri-
y^m existente $m < 2$, attamen $m > 1$. Sit igitur $m = 2 - \omega$ exi-
ctione unitate minore, ut sit $x = \alpha y^{2-\omega}$; erit $dx = \alpha(2-\omega)y^{1-\omega}dy$

$$\frac{dy}{dx} = \frac{1}{\alpha(2-\omega)y^{1-\omega}} = \infty$$

at radius osculi, qui subnormali $\frac{y\,dy}{dx}$ aequalis est, erit $= \frac{y^\omega}{\alpha(2-\omega)} = 0$.
vero curva radius osculi inventus est $= \frac{-1}{ly}$, unde radius osculi
n curva algebraica quacunque erit ad radium osculi in nostrae
to A ut $-y^\omega ly$ ad $\alpha(2-\omega)$, hoc est ut 0 ad 1; quantumvis
s sit exponens ω, casu $y = 0$ semper est $y^\omega ly = 0$, etiamsi sit
Quare in nostra quidem curva radius osculi in A est infinite
tamen infinities maior est quam radius osculi evanescens in omni
aica.

nito iam initio seriei, qua valor applicatae $PQ = q$ per abscissam
rimitur, scilicet

$$q = 1 - \frac{1}{2}pplp + App,$$

erit hinc formam totius seriei colligere. Cum enim ex aequa-
ntio-differentiali intelligatur sequentium terminorum potestates
ario crescere, valor ipsius q generatim gemina serie infinita ex-
tque

$$q = 1 + Ap^2 + Bp^4 + Cp^6 + Dp^8 + \text{etc.}$$
$$- \alpha pplp - \beta p^4lp - \gamma p^6lp - \delta p^8lp - \text{etc.},$$

em nunc iam novimus esse $\alpha = \frac{1}{2}$.

44. Cum igitur verus valor ipsius q duplici serie c
seorsim eliciamus, ponamus

$$q = r - slp$$

eritque differentiando

$$dq = dr - \frac{sdp}{p} - dslp, \quad ddq = ddr - \frac{2dpds}{p}$$

Hi valores in nostra aequatione differentiali

$$p(1-pp)ddq - dpdq(1+pp) + pqd$$

substituantur ac termini per lp affecti seorsim nihilo
duae obtinebuntur aequationes

$$\text{I.} \quad p(1-pp)dds - (1+pp)dpds + ps$$

$$\text{II.} \quad p(1-pp)ddr - (1+pp)dpdr + prdp^2 - 2(1-$$

45. Ad has aequationes resolvendas ponatur

$$r = 1 + Ap^2 + Bp^4 + Cp^6 + Dp^8 +$$

$$s = \alpha p^2 + \beta p^4 + \gamma p^6 + \delta p^8 + \varepsilon p^{10} +$$

eritque differentialibus sumendis

$$\frac{dr}{dp} = 2Ap + 4Bp^3 + 6Cp^5 + 8Dp^7$$

$$\frac{ddr}{dp^2} = 2A + 12Bp^2 + 30Cp^4 + 56Dp$$

$$\frac{ds}{dp} = 2\alpha p + 4\beta p^3 + 6\gamma p^5 + 8\delta p^7 +$$

$$\frac{dds}{dp^2} = 2\alpha + 12\beta p^2 + 30\gamma p^4 + 56\delta p^6$$

His valoribus substitutis prima aequatio abibit in ha

$$\begin{array}{rrrrrr}
 & 2\alpha p + & 12\beta p^3 + & 30\gamma p^5 + & 56\delta p^7 + & 90\varepsilon p^9 + \\
 & - 2\alpha & - 12\beta & - 30\gamma & - 56\delta & - \\
- 2\alpha & - 4\beta & - 6\gamma & - 8\delta & - 10\varepsilon & - \\
 & - 2\alpha & - 4\beta & - 6\gamma & - 8\delta & - \\
 & + \alpha & + \beta & + \gamma & + \delta & +
\end{array}$$

i iam singularum potestatum ipsius p coefficientes nihilo aequales erit

$$2\alpha - 2\alpha = 0; \qquad \alpha \text{ manet indeterminatum}$$

$$8\beta - 3\alpha = 0; \qquad \beta = \frac{1\cdot 3}{2\cdot 4}\alpha$$

$$24\gamma - 15\beta = 0; \qquad \gamma = \frac{3\cdot 5}{4\cdot 6}\beta = \frac{1\cdot 3\cdot 3\cdot 5}{2\cdot 4\cdot 4\cdot 6}\alpha$$

$$48\delta - 35\gamma = 0; \qquad \delta = \frac{5\cdot 7}{6\cdot 8}\gamma = \frac{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7}{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8}\alpha$$

$$80\varepsilon - 63\delta = 0; \qquad \varepsilon = \frac{7\cdot 9}{8\cdot 10}\delta = \frac{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9}{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdot 10}\alpha$$

$$\text{etc.} \qquad\qquad \text{etc.}$$

ur valor coefficientis primi α constaret, quem quidem iam vidimus omnes sequentes coefficientes β, γ, δ etc. forent cogniti. Verum alterius aequationis quoque hunc nobis valorem ipsius α pate-

bstitutis enim seriebus ante traditis in altera aequatione proveniet

$$\left.\begin{array}{rrrrrrr}
 & 2Ap & +12Bp^3 & +30Cp^5 & +56Dp^7 & +90Ep^9 & +\text{etc.} \\
 & & -2A & -12B & -30C & -56D & -\text{etc.} \\
 -2A & -4B & -6C & -8D & -10E & & -\text{etc.} \\
 & -2A & -4B & -6C & -8D & & -\text{etc.} \\
 +1 & +A & +B & +C & +D & & +\text{etc.} \\
 -4\alpha & -8\beta & -12\gamma & -16\delta & -20\varepsilon & & -\text{etc.} \\
 & +4\alpha & +8\beta & +12\gamma & +16\delta & & +\text{etc.} \\
 +2\alpha & +2\beta & +2\gamma & +2\delta & +2\varepsilon & & +\text{etc.}
\end{array}\right\} = 0.$$

$$48D - 35C - 14\delta + 12\gamma = 0; \qquad 6 \cdot 8\ D - 5 \cdot 7$$

$$80E - 63D - 18\varepsilon + 16\delta = 0; \qquad 8 \cdot 10E - 7 \cdot 9$$

etc.

48. Cognito igitur valore ipsius $\alpha = \frac{1}{2}$ altera
ipsius p involvit, tota innotescit; erit enim

$$\alpha = \frac{1}{2}$$

$$\beta = \frac{1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4}$$

$$\gamma = \frac{1 \cdot 1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6}$$

$$\delta = \frac{1 \cdot 1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8}$$

$$\varepsilon = \frac{1 \cdot 1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8}$$

etc.

fietque hinc

$$s = \alpha pp + \beta p^4 + \gamma p^6 + \delta p^8 +$$

49. Quod autem ad alteram seriem attinet

$$r = 1 + Ap^2 + Bp^4 + Cp^6 + Dp^8$$

primus coefficiens A hinc manet indeterminatu
has series ex aequatione differentiali secundi g
determinatione indiget, ut ad nostrum casum a

cientis A ex ipsa curvae natura definiri oportet, eo autem invento tescent ex his formulis, ad quas superiores redeunt:

$$B = \frac{1\cdot 3}{2\cdot 4}A - \frac{1}{8}\alpha\left(\frac{3}{2\cdot 2} + \frac{1}{1\cdot 1}\right)$$

$$C = \frac{3\cdot 5}{4\cdot 6}B - \frac{1}{8}\beta\left(\frac{3}{3\cdot 3} + \frac{1}{2\cdot 2}\right)$$

$$D = \frac{5\cdot 7}{6\cdot 8}C - \frac{1}{8}\gamma\left(\frac{3}{4\cdot 4} + \frac{1}{3\cdot 3}\right)$$

$$E = \frac{7\cdot 9}{8\cdot 10}D - \frac{1}{8}\delta\left(\frac{3}{5\cdot 5} + \frac{1}{4\cdot 4}\right)$$

etc.

s autem omnibus coefficientibus inventis ad datam quamvis abscis- p valor respondentis applicatae $PQ = q$ ita definitur, ut sit

$$\begin{aligned} q = 1 &+ Ap^2 + Bp^4 + Cp^6 + Dp^8 + \text{etc.} \\ &- \alpha pplp - \beta p^4lp - \gamma p^6lp - \delta p^8lp - \text{etc.}, \end{aligned}$$

, si abscissa p fuerit unitate multo minor, satis promte convergit, or ipsius q cognosci queat. Hinc vero etiam applicatae, quae ab- to maioribus unitate respondent, definiri poterunt, quia abscissae t applicata $\frac{q}{p}$. Quare si abscissa unitate multo maior ponatur respondens applicata $= Q$, ob $p = \frac{1}{P}$ et $q = pQ = \frac{Q}{P}$ erit

$$\begin{aligned} Q = P &+ AP^{-1} + BP^{-3} + CP^{-5} + DP^{-7} + \text{etc.} \\ &+ \alpha P^{-1}lP + \beta P^{-3}lP + \gamma P^{-5}lP + \delta P^{-7}lP + \text{etc.} \end{aligned}$$

scissa P fiat infinita, erit

$$Q = P + \frac{\alpha lP}{P} \quad \text{seu} \quad Q - P = \frac{\alpha lP}{P},$$

a rami Dq in infinitum extensi et ad asymtotam CV appropin- ligitur.

uia porro novimus, si $p = 1$, fore $q = \frac{\pi}{2}$, pro hoc casu aequatio nc formam ob $l1 = 0$ induet

$$\frac{\pi}{2} = 1 + A + B + C + D + E + \text{etc.}$$

Cum igitur valor A nondum sit definitus, reliqu
pendeant, haec aequatio conditionem continet,
natur. Ita scilicet valorem ipsius A comparatu
seriei infinitae $1 + A + B + C +$ etc. fiat $\frac{\pi}{2}$.
litterarum B, C, D etc., qui ab A pendent, evol
sultant expressiones, ut hinc valor ipsius A neut

52. Ad hanc constantem A determinandam
piam ellipsis perimeter ex altera formula in n
methodus cum requirat, ut omnes coefficientes
evolvantur, computo peracto reperietur

$\alpha = 0{,}5000000000;$	A quaeritur
$\beta = 0{,}1875000000;$	$B = 0{,}3750000000$
$\gamma = 0{,}1171875000;$	$C = 0{,}234375000$
$\delta = 0{,}0854492188;$	$D = 0{,}1708984375$
$\varepsilon = 0{,}0672912598;$	$E = 0{,}134582519$
$\zeta = 0{,}0555152893;$	$F = 0{,}111030578$
$\eta = 0{,}0472540855;$	$G = 0{,}0945081711$
$\theta = 0{,}0411363691;$	$H = 0{,}082272738$
$\iota = 0{,}0364228268;$	$I = 0{,}072845653$
$\varkappa = 0{,}0326793696;$	$K = 0{,}065358739$
	etc.

Hisque valoribus inventis, si abscissa sit C
definietur, ut sit

$$q = 1 + Ap^2 + Bp^4 + Cp^6 + Dp^8 + Ep^{10}$$
$$+ Ip^{18} + Kp^{20} + \text{et}$$
$$-pplp(\alpha + \beta p^2 + \gamma p^4 + \delta p^6 + \varepsilon p^8 + \zeta p^{10} + \eta p^1$$

1) Editio princeps: 0,0337966962. Correxit A.

de vero supra eiusdem applicatae q valorem ita invenimus ex-
t

$$\frac{1}{2}(1-pp)-\frac{1\cdot1\cdot1\cdot3}{2\cdot2\cdot4\cdot4}(1-pp)^2-\frac{1\cdot1\cdot1\cdot3\cdot3\cdot5}{2\cdot2\cdot4\cdot4\cdot6\cdot6}(1-pp)^3-\text{etc.}\Big)\cdot$$

ur ex utraque formula pro eodem quopiam valore ipsius p em ipsius q, ut deinceps ex aequalitate horum duorum elicere em coefficientis A. Pro p vero non nimis exiguam fractionem eniet, ne expressio posterior nimis lente convergat; tam parvum mus, ut coefficientes pro superiore forma computati valori q inveniendo sufficiant.

nus ergo ad commodum calculi $p=\frac{1}{5}$; erit in logarithmis hyper-

$$-lp=1{,}60943791243.$$

$$\begin{aligned}
\alpha pp &= 0{,}02000000000\\
\beta p^4 &= 0{,}00030000000\\
\gamma p^6 &= 0{,}00000750000\\
\delta p^8 &= 0{,}00000021875\\
\varepsilon p^{10} &= 0{,}00000000689\\
\zeta p^{12} &= 0{,}00000000023\\
\eta p^{14} &= 0{,}00000000001\\
\hline
& \ 0{,}02030772588 \quad \text{coefficiens ipsius } -lp\\
& \ 1{,}60943791243\\
& \ 0{,}03268402394 \quad \text{productum.}
\end{aligned}$$

$$\begin{aligned}
Ap^2 &= 0{,}04000000000\,A\\
Bp^4 &= 0{,}00060000000\,A - 0{,}00017500000\\
Cp^6 &= 0{,}00001500000\,A - 0{,}00000525000\\
Dp^8 &= 0{,}00000043750\,A - 0{,}00000016432\\
Ep^{10} &= 0{,}00000001378\,A - 0{,}00000000538\\
Fp^{12} &= 0{,}00000000045\,A - 0{,}00000000018\ ^{1)}\\
Gp^{14} &= 0{,}00000000002\,A - 0{,}00000000001\\
\hline
& \ 0{,}04061545175\,A - 0{,}00018041989\ ^{2)}
\end{aligned}$$

rinceps: 0,00000000016. 2) Editio princeps: 0,00018041987. Correxit A. K.

$$q = \frac{\pi}{2}\left(1 - \frac{1\cdot 1}{2\cdot 2}nn - \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}n^4 - \frac{1\cdot 1\cdot 1\cdot 3\cdot}{2\cdot 2\cdot 4\cdot 4\cdot}\right.$$

ponatur ad abbreviandum

$$q = \frac{\pi}{2} - \mathfrak{A}n^2 - \mathfrak{B}n^4 - \mathfrak{C}n^6 - \mathfrak{D}n^8 - \mathfrak{E}$$

Verum hoc casu ob $nn = \frac{24}{25}$ series ista nimis lente c
valor ipsius q satis exacte elici queat; quare, ut utr
tiam obtineamus, ponamus $p = \frac{1}{\sqrt{2}}$, ut sit tam $pp =$
culum vero tantum ad 6 figuras expediamus eritque

$$\begin{aligned}
App &= 0{,}500000\,A \\
Bp^4 &= 0{,}093750\,A - 0{,}027344 \\
Cp^6 &= 0{,}029297\,A - 0{,}010254 \\
Dp^8 &= 0{,}010681\,A - 0{,}004012 \\
Ep^{10} &= 0{,}004206\,A - 0{,}001640 \\
Fp^{12} &= 0{,}001735\,A - 0{,}000693 \\
Gp^{14} &= 0{,}000738\,A - 0{,}000300 \\
Hp^{16} &= 0{,}000321\,A - 0{,}000132 \\
Ip^{18} &= 0{,}000142\,A - 0{,}000059 \\
Kp^{20} &= 0{,}000064\,A - 0{,}000026 \\
\text{Summa reliquorum: } & 60\,A - \quad 24 \\
\text{Summa omnium } & 0{,}640994\,A - 0{,}044484 +
\end{aligned}$$

ergo

$$q = 1{,}066592 + 0{,}640994\,A,$$

1) Editio princeps: 1,03250360407. Correxit A. K.

pressio dat $q = 1{,}350647$, unde fit

$$A = \frac{284055}{640994} = 0{,}443147.$$

uamquam hic valor non ultra 6 figuras extenditur, tamen casui non videtur, quod iste numerus inventus 0,443147 a logarithmo bi-4718 unitatis quadrante 0,25 praecise deficiat. Quae coniectura sset consentanea, valorem litterae A ad plurimas figuras exhibere enim sit

$$l2 = 0{,}693147180559945309417232 1,$$

$2 - \frac{1}{4}$ ideoque

$$A = 0{,}4431471805599453094172321.$$

valor coefficientis huius A sit revera $= l2 - \frac{1}{4}$, sequenti mod ianeque coniecturam confirmo.

mparo scilicet arcum ellipticum AYP us semiaxes $AC = 1$, $CP = p$, cum olico AZS super eodem axe AC qui in A cum ellipsi communem vaturam. Sumta abscissa communi applicata ellipsis $XY = y$ et para-$= z$; erit

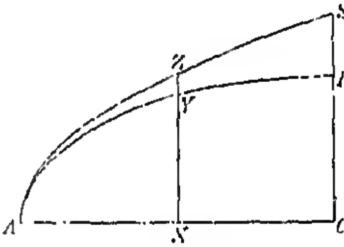


Fig. 2.

$$y = p\sqrt{(2x - xx)} \quad \text{et} \quad z = p\sqrt{2x}$$

$$dy = \frac{pdx(1-x)}{\sqrt{(2x - xx)}} \quad \text{et} \quad dz = \frac{pdx}{\sqrt{2x}},$$

cus ellipticus

$$AY = \int dx \sqrt{\left(1 + \frac{pp(1-x)^2}{2x - xx}\right)},$$

olicus

$$AZ = \int dx \sqrt{\left(1 + \frac{pp}{2x}\right)}.$$

Constat autem esse

$$AZ = x\sqrt{\left(1+\frac{pp}{2x}\right)} + \frac{1}{4}ppl\frac{\sqrt{\left(1+\frac{pp}{2x}\right)}+1}{\sqrt{\left(1+\frac{pp}{2x}\right)}-1}.$$

Hinc, si ponatur $x = 1$, erit arcus parabolicus

$$AZS = \sqrt{\left(1+\frac{1}{2}pp\right)} + \frac{1}{4}ppl\frac{\sqrt{\left(1+\frac{1}{2}pp\right)}+1}{\sqrt{\left(1+\frac{1}{2}pp\right)}-1}$$

At in formulis integralibus erit

$$\sqrt{\left(1+\frac{pp(1-x)^2}{2x-xx}\right)} = \sqrt{\left(1+\frac{pp}{2x} - \frac{pp(3-2x)}{4-2x}\right)}$$

Quia autem comparationem non ad altiores ipsius p po[t...] opus est quam ad secundam, coefficientes enim altiorum ips[i...] ex minoribus iam definivimus, reiectis terminis, qui contin[...] potestates, erit

$$\sqrt{\left(1+\frac{pp(1-x)^2}{2x-xx}\right)} = \sqrt{\left(1+\frac{pp}{2x}\right)} - \frac{pp(3-2x)}{4(2-x)}$$

ideoque

$$AY = \int dx\sqrt{\left(1+\frac{pp}{2x}\right)} - \frac{1}{4}pp\int\frac{dx(3-2x)}{2-x}$$

integralibusque actu sumtis

$$AY = x\sqrt{\left(1+\frac{pp}{2x}\right)} + \frac{1}{4}ppl\frac{\sqrt{\left(1+\frac{pp}{2x}\right)}+1}{\sqrt{\left(1+\frac{pp}{2x}\right)}-1} - \frac{1}{2}ppx - \frac{1}{4}$$

Ponatur iam $x = 1$, ut prodeat arcus $AYP = q$; erit

$$q = \sqrt{\left(1+\frac{1}{2}pp\right)} + \frac{1}{4}ppl\left(\sqrt{\left(1+\frac{1}{2}pp\right)}+1\right)$$
$$- \frac{1}{4}ppl\left(\sqrt{\left(1+\frac{1}{2}pp\right)}-1\right) - \frac{1}{2}pp + \frac{1}{4}ppl\dots$$

58. Iam quoniam ad altiores ipsius p potestates non re[s...]

$$\sqrt{\left(1+\frac{1}{2}pp\right)} = 1 + \frac{1}{4}pp,$$

fiet

$$q = 1 + \frac{1}{4}pp + \frac{1}{4}ppl\left(2 + \frac{1}{4}pp\right) - \frac{1}{4}ppl\frac{1}{4}pp - \frac{1}{2}pp + \frac{1}{4}ppl2.$$

pro $l\left(2 + \frac{1}{4}pp\right) = l2 + \frac{1}{8}pp$ scribere licet $l2$, ita ut sit

$$q = 1 - \frac{1}{4}pp + \frac{1}{2}ppl2 - \frac{1}{2}pplp + \frac{1}{2}ppl2$$

$$q = 1 - \frac{1}{2}pplp + pp\left(l2 - \frac{1}{4}\right).$$

o perspicitur coefficientem ipsius pp, quem ante littera A indicavi $= l2 - \frac{1}{4}$, omnino uti ex casu ante computato coniectura sumus conse

59. Pro curva igitur initio proposita $AQDq$ (Fig. 1, p. 22), si f cissa $CP = p$ et applicata $PQ = q$, erit

$$\begin{aligned} q = {} & 1 + App + Bp^4 + Cp^6 + Dp^8 + Ep^{10} + \text{etc.} \\ & -(\alpha pp + \beta p^4 + \gamma p^6 + \delta p^8 + \varepsilon p^{10} + \text{etc.})lp, \end{aligned}$$

coefficientes ita determinantur

$$\begin{aligned}
A &= l2 - \frac{1}{4}; & \alpha &= \frac{1}{2} \\
B &= \frac{1\cdot 3}{2\cdot 4}A - \frac{1}{2}(\alpha - \beta) + \frac{1}{2}\cdot\frac{\beta}{2}; & \beta &= \frac{1\cdot 3}{2\cdot 4}\alpha \\
C &= \frac{3\cdot 5}{4\cdot 6}\cdot B - \frac{1}{3}(\beta - \gamma) + \frac{1}{4}\cdot\frac{\gamma}{3}; & \gamma &= \frac{3\cdot 5}{4\cdot 6}\beta \\
D &= \frac{5\cdot 7}{6\cdot 8}C - \frac{1}{4}(\gamma - \delta) + \frac{1}{6}\cdot\frac{\delta}{4}; & \delta &= \frac{5\cdot 7}{6\cdot 8}\gamma \\
E &= \frac{7\cdot 9}{8\cdot 10}D - \frac{1}{5}(\delta - \varepsilon) + \frac{1}{8}\cdot\frac{\varepsilon}{5}; & \varepsilon &= \frac{7\cdot 9}{8\cdot 10}\delta \\
F &= \frac{9\cdot 11}{10\cdot 12}E - \frac{1}{6}(\varepsilon - \zeta) + \frac{1}{10}\cdot\frac{\zeta}{6}; & \zeta &= \frac{9\cdot 11}{10\cdot 12}\varepsilon
\end{aligned}$$

etc.

Series haec valde convergit, si abscissa p fueri
sit unitate multo maior, iisdem manentibus co

$$q = p + \frac{A}{p} + \frac{B}{p^3} + \frac{C}{p^5} + \frac{D}{p^7}$$

$$+ \left(\frac{\alpha}{p} + \frac{\beta}{p^3} + \frac{\gamma}{p^5} + \frac{\delta}{p^7} + \right.$$

60. **Verum** si abscissa p non multum ab
hac serie supra § 26 inventa

$$q = 1 + pp \left\{ \begin{aligned} &\left(\frac{\pi}{2} - 1\right) + \left(\frac{1 \cdot 3}{2 \cdot 2} \cdot \frac{\pi}{2} - 1\right)(1 - pp) \\ &\qquad + \left(\frac{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} \cdot \frac{\pi}{2} - \right. \end{aligned} \right.$$

quae etiam ex natura ellipsis in hanc convert

$$q = p + \frac{1}{p} \left\{ \begin{aligned} &\left(\frac{\pi}{2} - 1\right) - \left(\frac{1 \cdot 3}{2 \cdot 2} \cdot \frac{\pi}{2} - 1\right) \frac{(1 - pp)}{pp} \\ &\qquad - \left(\frac{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} \cdot \frac{\pi}{2} - \right. \end{aligned} \right.$$

de, prout fuerit vel $p > 1$ vel $p < 1$, eam
dem signis procedant vel alternantibus.
mmam proxime definiendam signa eligere a

PROBLEMA

61. *Datis axibus coniugatis ellipsis in nu*
etrum.

SOLUTIO

Sint semiaxes ellipsis 1 et p et quad
rmulas inventas valor ipsius q in numeri
igatur, cuius termini maxime convergant.
rmulas, quae sunt

$$\text{I.} \quad q = 1 + App + Bp^4 + Cp^6 + Dp^8$$

$$- (\alpha pp + \beta p^4 + \gamma p^6 + \delta p^8 + \varepsilon p$$

$$q = p + A\frac{1}{p} + B\frac{1}{p^3} + C\frac{1}{p^5} + D\frac{1}{p^7} + E\frac{1}{p^9} + F\frac{1}{p^{11}} + \text{etc.}$$
$$+\left(\frac{\alpha}{p} + \frac{\beta}{p^3} + \frac{\gamma}{p^5} + \frac{\delta}{p^7} + \frac{\varepsilon}{p^9} + \frac{\zeta}{p^{11}} + \text{etc.}\right) lp$$

$$pp\left(\mathfrak{A} + \mathfrak{B}(1-pp) + \mathfrak{C}(1-pp)^2 + \mathfrak{D}(1-pp)^3 + \mathfrak{E}(1-pp)^4 + \text{etc.}\right)$$

$$\frac{1}{p}\left(\mathfrak{A} - \mathfrak{B}\frac{(1-pp)}{pp} + \mathfrak{C}\frac{(1-pp)^2}{p^4} - \mathfrak{D}\frac{(1-pp)^3}{p^6} + \mathfrak{E}\frac{(1-pp)^4}{p^8} - \text{etc.}\right).$$

autem tergeminorum coefficientium valores sunt in numeris

0,44314718056	$\alpha = 0{,}50000000000$	$\mathfrak{A} = 0{,}57079632679$
0,05680519271	$\beta = 0{,}18750000000$	$\mathfrak{B} = 0{,}17809724510$
0,02183137044	$\gamma = 0{,}11718750000$	$\mathfrak{C} = 0{,}10446616728$
0,01154452144 [1]	$\delta = 0{,}08544921875$	$\mathfrak{D} = 0{,}07378655152$
0,00714200029	$\varepsilon = 0{,}06729125977$	$\mathfrak{E} = 0{,}05700863665$
0,00485474337	$\zeta = 0{,}05551528931$ [2]	$\mathfrak{F} = 0{,}04643855029$
0,00351468795	$\eta = 0{,}04725408554$	$\mathfrak{G} = 0{,}03917161591$
0,00266223578	$\theta = 0{,}04113636911$	$\mathfrak{H} = 0{,}03386971991$
0,00208639732	$\iota = 0{,}03642282682$	$\mathfrak{I} = 0{,}02983116632$
0,00167916842	$\varkappa = 0{,}03267936962$	$\mathfrak{K} = 0{,}02665267507$
		$\mathfrak{L} = 0{,}02408604338$ [3]

ro quavis ellipsis specie habebitur series convergens, unde eius finiri poterit; veluti si ponatur

$$p = \frac{1}{10}, \quad \text{erit} \quad q = 1{,}015993545021,$$

$$p = \frac{1}{5}, \quad \text{erit} \quad q = 1{,}05050222700,$$

$$p = \frac{1}{\sqrt{2}}, \quad \text{erit} \quad q = 1{,}3506429.$$

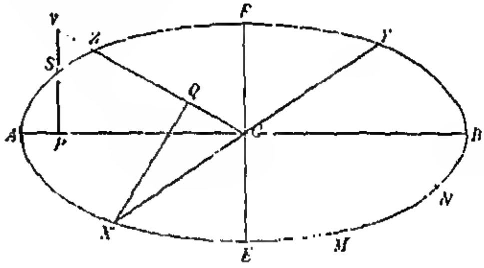
princeps: 0,01154452143. 2) Editio princeps: 0,05551527931. 3) Editio 08604339. Correxit A. K.

PROBLEMA AD CUIUS SOLUTIO
GEOMETRAE INVITANTU
THEOREMA AD CUIUS DEMONSTR
GEOMETRAE INVITANTU

Commentatio 211 indicis ENESTROEMIANI
Nova acta eruditorum 1754, p. 40

PROBLEMA
AD CUIUS SOLUTIONEM GEOMETRAE INV

Proposito quadrante elliptico BNME inter binos sem et CE intercepto in eo geometrice assignare puncta M et N cise sit semissis arcus quadrantis BNME.



THEOREMA
AD CUIUS DEMONSTRATIONEM GEOMETRAE

Si ellipsis AEBF axibus principalibus AB et EF d quamcumque obliquam XCY bissecetur, ad quam semidiame ducatur in V, ut sit CV aequalis semiaxi CA, et ex V a

...ellipsi ab altero in puncto B, ut arcuum XAS et YFS different... assignari possit.

...nim ex X ad CZ perpendiculum XQ ducatur, intervallum CQ t... quale erit illorum arcuum differentiae seu erit

$$YFS - XAS = 2CQ.$$

...ifficilius autem tam Problema resolvendum videtur quam Theorem... ndum, quod diversi arcus elliptici nullo adhuc modo inter se co... tuerint, unde ex harum propositionum pertractatione non conten... alyseos incrementa merito exspectantur. Graviore autem praem... ad hoc argumentum suscipiendum incitari non possunt.

...problematis et demonstratio theorematis inveniuntur in L. Euleri Commentatione 2... ...ENESTROEMIANO); vide p. 291 A. K.

DE INTEGRATIONE AEQUATIONIS DIFFE

$$\frac{m\,dx}{\sqrt{(1-x^4)}} = \frac{n\,dy}{\sqrt{(1-y^4)}}$$

Commentatio 251 indicis ENESTROEMIANI
Novi commentarii academiae scientiarum Petropolitanae 6 (1756/7), 1
Summarium ibidem p. 7—9

SUMMARIUM

In hac dissertatione et nonnullis sequentibus, quibus simile argu
quasi novus plane campus in Analysi aperitur integralia diversarum fo
se omnem integrationis solertiam respuunt, inter se comparandi. Cum
parationis angulorum relatio inter binas variabiles x et y huic aequati

$$\frac{m\,dx}{\sqrt{(1-xx)}} = \frac{n\,dy}{\sqrt{(1-yy)}}$$

conveniens algebraice exhiberi queat, etsi utraque formula per se algebr
sed angulum seu arcum circularem exprimit, haec relatio ex eo tantum
quod angulorum datam et quidem rationalem rationem tenentium sinu
comparari possunt. Neque talis comparatio locum habere videtur, nisi
per angulos sive per logarithmos integrari queant. Quoties quidem so
blematis ad huiusmodi aequationem differentialem $Xdx = Ydy$, in
ipsius x et Y ipsius y, tantum perducitur, ea, quia variabiles sunt a
tanquam penitus absoluta spectari solet, cum ope quadraturae duarum
alterius area per $\int Xdx$, alterius per $\int Ydy$ exprimitur, construi p
dato quovis valore ipsius x valor ipsius y conveniens assignari debe
draturam involvere videtur, sine qua relatio inter x et y minime e
magis igitur mirum videbitur, cum talis formulae $\frac{dz}{\sqrt{(1-z^4)}}$ integral
neque per logarithmos exprimi possit, quae quantitates transcendent
solae idoneae putantur, nihilominus pro aequatione differentiali propo

algebraice exhiberi posse, ita ut linea curva, cuius arcus indefinite hac formula int
$\frac{d}{..x}$ exprimitur, pari proprietate ac circulus sit praedita, ut scilicet omnes eius
ac comparari seu proposito in eo arcu quocumque alius arcus, qui ad eum r
d rationem, geometrice assignari queat. Vel, quod eodem redit, aequatio inte
ationis differentialis propositae, quae veram relationem inter x et y exprimit,
m non tale integrale involvet, sed adeo erit algebraica.

Atque hoc quidem non tantum pro casu quodam particulari, verum adeo inde
definitum, quod quantitatem constantem arbitrariam complectitur, erit algebraicum. N
talis admiranda integratio in ipsa tantum aequatione differentiali locum habet
i omnino modo Cel. Auctor ostendit hanc aequationem differentialem multo latius pat

$$\frac{m\,dx}{\sqrt{(A + Bx^2 + Cx^4)}} = \frac{n\,dy}{\sqrt{(A + By^2 + Cy^4)}}$$

aequationem algebraicam complete integrari posse, si modo numeri m et n sint
s; quin etiam eandem integrandi methodum ad hanc aequationem multo generali
ndit

$$\frac{m\,dx}{\sqrt{(A + Bx + Cx^2 + Dx^3 + Ex^4)}} = \frac{n\,dy}{\sqrt{(A + By + Cy^2 + Dy^3 + Ey^4)}}$$

in denominatoribus radicalibus *omnes potestates ipsarum x et y ad quartam usq*
unt. Hinc suspicari liceret, etiamsi hae potestates ultra ascenderent, integrat
m *algebraicam adhuc locum esse habituram; sed praeterquam quod methodus Au*
on potestate quarta terminatur, facile ostendi potest, in potestate certe sexta algebr
grationem in genere excludi. Si enim coefficientes ita accipiantur, ut radix qu
ahi queat, ex hoc solo casu $\frac{m\,dx}{1+x^2} = \frac{n\,dy}{1+y^2}$ evidens est relationem inter x et y
n algebraice exprimi posse, cum utrinque formulae integrale tam angulum quam
hmum involvat; anguli autem et logarithmi certe inter se algebraice comparari n
tur. Interim tamen peculiari modo integratio huius quoque aequationis

$$\frac{m\,dx}{\sqrt{(A + Bx^2 + Cx^4 + Dx^6)}} = \frac{n\,dy}{\sqrt{(A + By^2 + Cy^4 + Dy^6)}}$$

braice exhibetur, unde patet hanc dissertationem multo plures investigationes con
n titulus quidem prae se ferre videtur.

1. *Cum primum occasione inventionum* Ill. Comitis FAGNANI[1]) hanc a
nem essem contemplatus, eiusmodi quidem relationem algebraicam

1) G. C. FAGNANO (1682—1766), *Produzioni matematiche*, T. 2, Pesaro 1750; *Oper*
che, T. 2, Milano Roma Napoli 1911. A. K.

aequatione integrali completa haberi poterat, propterea quod
retur quantitatem constantem arbitrariam, cuiusmodi semper
integrationem introduci solet. Hinc enim, uti satis notum
completa et particularia distingui solent, quorum illa totam
differentialium exhauriunt, haec vero tantum ita satisfaciunt,
expressiones aeque satisfacere queant. Criterium autem aeq
completae in hoc consistit, quod ea quantitatem constantem
quae in aequatione differentiali non apparet.

2. Quae quo clarius perspiciantur, sufficiet aequatione
simplicissimam $dx = dy$ considerasse, cui utique satisfacit hac
in rem tamen haec integralis minus late patet quam diffe
cum huic aeque satisfaciat haec integralis $x = y \mp a$ mul
sumendo pro a quantitatem constantem quamcunque, atque
gralis totam vim aequationis differentialis $dx = dy$ exhaurire
etiam aequatio integralis completa appellatur, propterea
quantitas constans a, quae in aequatione differentiali non
vero loco istius constantis indefinitae a valores determinati
integrali completo obtinentur integralia particularia, qua
rationem minus late patent, quam aequatio differentialis pr

3. Saepe numero autem aequationis differentialis int
algebraicum exhiberi potest, cum tamen integrale completum
hoc scilicet evenit, si pars transcendens per constantem
fuerit multiplicata, quae propterea constante illa nihilo
calculo evanescit et integrale algebraicum particulare re
aequationi $dy = dx + (y - x)dx$ manifestum est satisfacer
quo tamen tantum integrale particulare continetur, cu
$y = x + ae^x$ denotante e numerum, cuius logarithmus est
constans arbitraria a evanescens ponatur, integrale semper

4. Cum igitur evenire queat, ut aequatio differential
culare algebraicum admittat, etiamsi integrale completum
ita etiam rationes dubitandi non desunt, quod integrale com
differentialis propositae

$$\frac{m\,dx}{\sqrt{(1-x^4)}} = \frac{n\,dy}{\sqrt{(1-y^4)}}$$

uantitates transcendentes involvat, etiamsi pro ea integrale particulare a aicum exhibere licuerit. Cum enim integrale completum sit

$$m\int\frac{dx}{\sqrt{(1-x^4)}} = n\int\frac{dy}{\sqrt{(1-y^4)}} + C,$$

ec autem integralia nullo modo, neque circuli neque hyperbolae qua ram in subsidium vocando, assignari queant, minime probabile videtur i rmulas tantopere transcendentes in genere, ita ut constans C maneat terminata, ad relationem algebraicam inter x et y revocari posse.

5. Notum quidem est integrale completum huius aequationis differenti

$$\frac{m\,dx}{\sqrt{(1-xx)}} = \frac{n\,dy}{\sqrt{(1-yy)}}$$

mper algebraice exhiberi posse, dummodo proportio coefficientium m e erit rationalis; sed quia utriusque formulae integrale arcum circuli indi ut integrale completum sit $m\,\text{A}\sin. x = n\,\text{A}\sin. y + C$, relatio autem um, qui ad arcus proportionem rationalem inter se tenentes spectant, a aice exprimi potest, mirum non est aequationem integralem completam sibus quoque algebraice exhiberi posse. Cum autem huiusmodi comparatio rmulis transcendentibus $\int\frac{dx}{\sqrt{(1-x^4)}}$ et $\int\frac{dy}{\sqrt{(1-y^4)}}$ locum non habeat seu salt n constet, inde reductio integralis ad quantitates algebraicas peti non pote

6. Nihilo tamen minus observavi, si proposita fuerit huiusmodi aequa ferentialis

$$\frac{m\,dx}{\sqrt{(1-x^4)}} = \frac{n\,dy}{\sqrt{(1-y^4)}},$$

am integrale completum, quod scilicet quantitatem constantem arbitrari volvat, semper algebraice exprimi posse, dummodo ratio $m:n$ fuerit ra lis; quod mihi quidem eo magis notatu dignum videtur, quod nulla c ethodo ad hoc integrale sum perductus, sed id potius tentando vel d ndo eliciι. Unde nullum est dubium, quin methodus directa ad idem tegrale perducens fines Analyseos non mediocriter sit amplificatura; cu optorea investigatio Analystis omni studio commendanda videtur.

fuerit ratio rationalis coefficientium m et n, derivare mihi licui
tione completa huius aequationis

$$\frac{dx}{\sqrt{(1-x^4)}} = \frac{dy}{\sqrt{(1-y^4)}};$$

hac enim concessa methodum certam indicabo ex ea quoque i
pletum huius aequationis multo latius patentis

$$\frac{m\,dx}{\sqrt{(1-x^4)}} = \frac{n\,dy}{\sqrt{(1-y^4)}}$$

concludendi. Quae methodus etiam in genere ad huiusmodi
$mXdx = nYdy$ integralia invenienda adhiberi queat, si modo i
pletum huius $Xdx = Ydy$ fuerit erutum atque Y talem signific
ipsius y, qualis X est ipsius x.

8. Exordiar igitur ab hac aequatione

$$\frac{dx}{\sqrt{(1-x^4)}} = \frac{dy}{\sqrt{(1-y^4)}},$$

cui quidem primo intuitu satisfacere perspicuum est aequationem
propterea eius est integrale particulare. Tum vero eidem aequ
satisfacit iste valor algebraicus

$$x = -\sqrt{\frac{1-yy}{1+yy}};$$

cum enim sit

$$dx = +\frac{2y\,dy}{(1+yy)\sqrt{(1-yy)(1+yy)}} \quad \text{et} \quad \sqrt{(1-x^4)} = \frac{2}{1+}$$

erit

$$\frac{dx}{\sqrt{(1-x^4)}} = \frac{dy}{\sqrt{(1-y^4)}}.$$

Hinc iste etiam valor seu aequatio $xxyy + xx + yy - 1 = 0$
particularis aequationis differentialis propositae. Unde integral
quod constantem arbitrariam involvat, ita comparatum sit n
tribuendo huic constanti certum quendam valorem prodeat

$$x = y,$$

sin autem eidem constanti alius quidem valor tribuatur, ut pro

$$x = -\sqrt{\frac{1-yy}{1+yy}} \quad \text{seu} \quad xxyy + xx + yy - 1 = 0.$$

THEOREMA

9. *Dico igitur huius aequationis differentialis*

$$\frac{dx}{\sqrt{(1-x^4)}} = \frac{dy}{\sqrt{(1-y^4)}}$$

aequationem integralem completam esse

$$xx + yy + ccxxyy = cc + 2xy\sqrt{(1-c^4)}.$$

DEMONSTRATIO

Posita enim hac aequatione eius differentiale erit

$$xdx + ydy + ccxy(xdy + ydx) = (xdy + ydx)\sqrt{(1-c^4)},$$

unde fit

$$dx(x + ccxyy - y\sqrt{(1-c^4)}) + dy(y + ccxxy - x\sqrt{(1-c^4)}) = 0.$$

Ex eadem vero aequatione resoluta colligitur

$$y = \frac{x\sqrt{(1-c^4)} + c\sqrt{(1-x^4)}}{1 + ccxx} \quad \text{et} \quad x = \frac{y\sqrt{(1-c^4)} - c\sqrt{(1-y^4)}}{1 + ccyy}.$$

Si enim ibi radicali $\sqrt{(1-x^4)}$ tribuitur signum $+$, hic radicali $\sqrt{(1-y^4)}$ signum $-$ tribui debet, ut posito $x = 0$ utrinque idem valor prodeat $y = c$. Erit ergo

$$x + ccxyy - y\sqrt{(1-c^4)} = -c\sqrt{(1-y^4)},$$

$$y + ccxxy - x\sqrt{(1-c^4)} = \quad c\sqrt{(1-x^4)},$$

quibus valoribus in aequatione differentiali substitutis prodit

$$-cdx\sqrt{(1-y^4)} + cdy\sqrt{(1-x^4)} = 0$$

seu

$$\frac{dx}{\sqrt{(1-x^4)}} = \frac{dy}{\sqrt{(1-y^4)}}.$$

Huius ergo aequationis differentialis integrale est

$$xx + yy + ccxxyy = cc + 2xy\sqrt{(1-c^4)},$$

quod, quia constantem c ab arbitrio nostro pendentem continet, erit simul integrale completum. Q. E. D.

10. Si igitur habeatur haec aequatio $\sqrt{(1-x^4)} = \sqrt{(1-y^4}$
completus ipsius x est

$$x = \frac{y\sqrt{(1-c^4)} \pm c\sqrt{(1-y^4)}}{1+ccyy},$$

unde, si constans arbitraria c evanescat, fit $x = y$; sin aute
habemus $x = \pm\frac{\sqrt{(1-y^4)}}{1+yy} = \sqrt{\frac{1-yy}{1+yy}}$, qui sunt ambo illi va
iam supra exhibiti. Hinc eruuntur alii valores particula
simpliciores, sed qui ad imaginaria devolvuntur. Ita posito

$$x = \frac{\sqrt{-1}}{y}$$

et posito $cc = -1$ fit

$$x = \sqrt{\frac{yy+1}{yy-1}},$$

qui itidem aequationi propositae satisfaciunt.

11. Quo autem ratio huius integralis clarius perspic
curva AM (Fig. 1), cuius haec sit indoles, ut posita abscissa

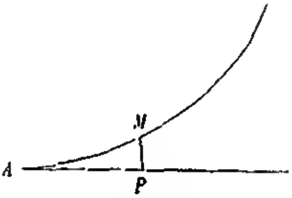


Fig. 1.

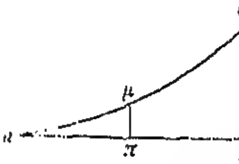


Fig. 2.

ei respondens $AM = \int \frac{du}{\sqrt{(1-u^4)}}$. Deinde eadem curva denu
capiatur abscissa $ap = x$; erit arcus $am = \int \frac{dx}{\sqrt{(1-x^4)}}$. Su

$$x = \frac{u\sqrt{(1-c^4)} \pm c\sqrt{(1-u^4)}}{1+ccuu}$$

fiet $\frac{dx}{\sqrt{(1-x^4)}} = \frac{du}{\sqrt{(1-u^4)}}$ ideoque arc. am = arc. AM + Cons

huius determinatione posito $u = 0$, quo casu arcus AM evanescit
Quare si capiatur abscissa $ab = c$, cui arcus ad respondeat, erit a
arcui AM.

2. Ope huius ergo integrationis completae aequationis $\frac{dx}{\sqrt{(1-x^4)}} = \frac{d}{\sqrt{(1}}$
va proposita arcui cuicumque AM, qui abscissae $AP = u$ respon
aequalis dm, qui a dato puncto d incipiat, abscindi poterit. P
bscissa dato puncto d respondente $ab = c$ si capiatur abscissa

$$ap = x = \frac{c\sqrt{(1-u^4)} + u\sqrt{(1-c^4)}}{1+ccuu},$$

rcus dm arcui AM aequalis. Simili autem modo cum $\sqrt{(1-c^4)}$ n
statui liceat, si capiatur abscissa

$$a\pi = \frac{c\sqrt{(1-u^4)} - u\sqrt{(1-c^4)}}{1+ccuu},$$

idem arcus $d\mu$ arcui AM aequalis sicque in hac curva a dato qu
d utrinque abscindi potest arcus dm et $d\mu$, qui arcui AM
os.

3. Hinc ergo patet, si arcus ad aequalis capiatur arcui AM seu c
rcum am duplum arcus AM. Hinc si statuatur $ap = x = \frac{2u\sqrt{(1-}}{1+}$
it arcus $am = 2$ arc. AM. Simili modo si capiatur arcus $ad = 2$
$= \frac{2u\sqrt{(1-u^4)}}{1+u^4}$ statuaturque $x = \frac{c\sqrt{(1-u^4)} + u\sqrt{(1-c^4)}}{1+ccuu}$, obtinebitur a
3 arc. AM. Ac si iste valor ipsius x denuo pro c substituatur
$= 3AM$, iterumque statuatur $x = \frac{c\sqrt{(1-u^4)} + u\sqrt{(1-c^4)}}{1+ccuu}$, nascetur a
uadruplus arcus AM; atque ita porro successive quaecumque mul
AM geometrice assignari poterunt.

4. Sit arcus $ad = n \cdot AM$ et $ab = z$, ita ut sit

$$\int \frac{dz}{\sqrt{(1-z^4)}} = n \int \frac{du}{\sqrt{(1-u^4)}};$$

atque ex his patet, si capiatur

$$x = \frac{z\sqrt{(1-u^4)}+u\sqrt{(1-z^4)}}{1+uuzz},$$

fore

$$\int\frac{dx}{\sqrt{(1-x^4)}} = (n+1)\int\frac{du}{\sqrt{(1-u^4)}};$$

sin autem ponatur

$$x = \frac{z\sqrt{(1-u^4)}-u\sqrt{(1-z^4)}}{1+uuzz},$$

tum futurum esse

$$\int\frac{dx}{\sqrt{(1-x^4)}} = (n-1)\int\frac{du}{\sqrt{(1-u^4)}}.$$

Si igitur haec aequatio $\frac{dz}{\sqrt{(1-z^4)}} = \frac{ndu}{\sqrt{(1-u^4)}}$ fuerit integrata deb pro z inde erutus, etiam integrari poterit haec aequatio $\frac{dx}{\sqrt{(1-x}}$ quippe cuius integrale erit $x = \frac{z\sqrt{(1-u^4)}\pm u\sqrt{(1-z^4)}}{1+uuzz}$. Ac si pro fuerit eius valor completus, qui scilicet constantem arbitrariam in pro x prodibit eius valor completus.

15. Hinc igitur perspicuum est, quomodo aequatio integralis veniri debeat, quae conveniat huic aequationi differentiali $\frac{dx}{\sqrt{(1-x}}$ quoties n fuerit numerus integer. Simili autem modo assign ut sit $\frac{dy}{\sqrt{(1-y^4)}} = \frac{mdu}{\sqrt{(1-u^4)}}$; unde, si eliminando u aequatio inter ratur, ea erit integralis huius aequationis $\frac{mdx}{\sqrt{(1-x^4)}} = \frac{ndy}{\sqrt{(1-y}}$ numeri rationales pro m et n substituantur; atque ut hoc int completum, sufficit pro altera tantum variabilium x et y valor per u determinasse, cum hinc iam nova constans arbitraria introducatur.

16. Methodus, qua hic in theorematis demonstratione sum ex rei natura est petita, sed indirecte ad id, quod propositum tamen multo latius patet; simili enim modo colligitur huiu differentialis

$$\frac{dx}{\sqrt{(1+mxx+nx^4)}} = \frac{dy}{\sqrt{(1+myy+ny^4)}}$$

integrale completum esse

$$0 = cc - xx - yy + ncc xxyy + 2xy\sqrt{(1 + mcc + nc^4)}.$$

Unde idem quod ante ratiocinium adhibendo integrale quoque com obtinebitur huius aequationis

$$\frac{\mu dx}{\sqrt{(1 + mxx + nx^4)}} = \frac{\nu dy}{\sqrt{(1 + myy + ny^4)}},$$

siquidem litteris μ et ν numeri integri designentur.

17. Investigatio autem huius integrationis ita se habet: Fingatur pro arbitrio relatio inter variabiles x et y hac aequatione contenta

$$(1) \quad \alpha xx + \alpha yy = 2\beta xy + \gamma xxyy + \delta,$$

quae differentiata dat

$$\alpha xdx + \alpha ydy = \beta xdy + \beta ydx + \gamma xyydx + \gamma xxydy,$$

unde conficitur

$$(2) \quad dx(\alpha x - \beta y - \gamma xyy) + dy(\alpha y - \beta x - \gamma xxy) = 0.$$

Deinde ex aequatione (1) eliciantur valores utriusque variabilis

$$x = \frac{\beta y + \sqrt{(\alpha\delta + (\beta\beta - \alpha\alpha - \gamma\delta)yy + \alpha\gamma y^4)}}{\alpha - \gamma yy},$$

$$y = \frac{\beta x - \sqrt{(\alpha\delta + (\beta\beta - \alpha\alpha - \gamma\delta)xx + \alpha\gamma x^4)}}{\alpha - \gamma xx}.$$

Atque hinc obtinemus

$$(3) \quad \alpha x - \beta y - \gamma xyy = \sqrt{(\alpha\delta + (\beta\beta - \alpha\alpha - \gamma\delta)yy + \alpha\gamma y^4)},$$

$$(4) \quad \alpha y - \beta x - \gamma xxy = -\sqrt{(\alpha\delta + (\beta\beta - \alpha\alpha - \gamma\delta)xx + \alpha\gamma x^4)},$$

qui valores in aequatione (2) substituti praebebunt

$$(5) \quad \frac{dx}{\sqrt{(\alpha\delta + (\beta\beta - \alpha\alpha - \gamma\delta)xx + \alpha\gamma x^4)}} = \frac{dy}{\sqrt{(\alpha\delta + (\beta\beta - \alpha\alpha - \gamma\delta)yy + \alpha\gamma y^4)}},$$

cuius ergo aequationis integrale est aequatio (1).

18. Quo istas formas simpliciores reddamus, ponamus

$$\alpha\delta = A, \quad \beta\beta - \alpha\alpha - \gamma\delta = C, \quad \alpha\gamma = E$$

eritque

$$\delta = \frac{A}{\alpha}, \quad \gamma = \frac{E}{\alpha} \quad \text{et} \quad \beta = \sqrt{\left(C + \alpha\alpha + \frac{AE}{\alpha\alpha}\right)}.$$

Quare huius aequationis differentialis

$$(6) \quad \frac{dx}{\sqrt{(A + Cxx + Ex^4)}} = \frac{dy}{\sqrt{(A + Cyy + Ey^4)}}$$

aequatio integralis est haec

$$(7) \quad \alpha(xx + yy) = \frac{A}{\alpha} + \frac{E}{\alpha}xxyy + 2xy\sqrt{\left(C + \alpha\alpha + \frac{AE}{\alpha\alpha}\right)},$$

quae simul est integralis completa.

19. Vel ponamus

$$A = f\alpha\alpha, \quad C = g\alpha\alpha \quad \text{et} \quad E = h\alpha\alpha,$$

ut habeamus hanc aequationem differentialem

$$\frac{dx}{\sqrt{(f + gxx + hx^4)}} = \frac{dy}{\sqrt{(f + gyy + hy^4)}},$$

cuius propterea aequatio integralis completa erit

$$xx + yy = f + hxxyy + 2xy\sqrt{(1 + g + fh)};$$

quae etsi novam constantem involvere non videtur, tamen est c… in differentiali tantum ratio quantitatum f, g et h spectetur, … g et h scribere liceat fcc, gcc et hcc, unde aequatio integra… completa prodit

$$xx + yy = fcc + hccxxyy + 2xy\sqrt{(1 + gcc + fhc^4)}$$

vel

$$f(xx + yy) = fee + heexxyy + 2xy\sqrt{f(f + gee + he^4)}$$

posito $cc = \frac{ee}{f}$.

Quodsi ergo proposita sit haec aequatio differentialis

$$\frac{dx}{\sqrt{(f+gxx+hx^4)}} = \frac{dy}{\sqrt{(f+gyy+hy^4)}},$$

...ius y per functionem algebraicam ipsius x exprimi poterit, ita ut sit

$$y = \frac{x\sqrt{(1+gee+fhe^4)}+e\sqrt{(1+gxx+fhx^4)}}{1-heexx}$$

$$y = \frac{x\sqrt{f(f+gee+he^4)}+e\sqrt{f(f+gxx+hx^4)}}{f-heexx}.$$

...dsi ergo fit $y = 0$, ut habeatur haec aequatio differentialis

$$\frac{dx}{\sqrt{(f+hx^4)}} = \frac{dy}{\sqrt{(f+hy^4)}},$$

...egralis completus ipsius y erit

$$y = \frac{x\sqrt{f(f+he^4)}+e\sqrt{f(f+hx^4)}}{f-heexx},$$

...nstantem e pro lubitu determinando innumeri valores particulares ...duci possunt.

Methodi autem, qua supra usi sumus, beneficio etiam huius aequationis

$$\frac{mdx}{\sqrt{(f+gxx+hx^4)}} = \frac{ndy}{\sqrt{(f+gyy+hy^4)}},$$

... m et n sint numeri rationales, integrale completum atque id quidem ...e exhiberi poterit.

Quemadmodum in aequatione supra assumta variabiles x et y inter se ...iles sunt constitutae, ut ambae formulae inter se similes evaderent, ...ssa hac limitatione ad formularum differentialium disparium com... ...m perveniemus. Ponamus ergo

$$(1) \quad \alpha xx + \beta yy - 2\gamma xy + \delta xxyy + \varepsilon,$$

unde fit

$$x = \frac{\gamma y + \sqrt{(\alpha\varepsilon + (\gamma\gamma - \delta\varepsilon - \alpha\beta)yy + \beta\delta y^4)}}{\alpha - \delta yy}$$

et

$$y = \frac{\gamma x - \sqrt{(\beta\varepsilon + (\gamma\gamma - \delta\varepsilon - \alpha\beta)xx + \alpha\delta x^4)}}{\beta - \delta xx}$$

hincque

$$(2) \quad \alpha x - \gamma y - \delta xyy = \sqrt{(\alpha\varepsilon + (\gamma\gamma - \delta\varepsilon - \alpha\beta)\, yy}$$

$$(3) \quad \beta y - \gamma x - \delta xxy = -\sqrt{(\beta\varepsilon + (\gamma\gamma - \delta\varepsilon - \alpha\beta)}$$

at aequatio (1) differentiata dat

$$dx(\alpha x - \gamma y - \delta xyy) + dy(\beta y - \gamma x - \delta xxy) =$$

unde conficitur haec aequatio differentialis

$$\frac{dx}{\sqrt{(\beta\varepsilon + (\gamma\gamma - \delta\varepsilon - \alpha\beta)xx + \alpha\delta x^4)}} = \frac{dy}{\sqrt{(\alpha\varepsilon + (\gamma\gamma - \delta\varepsilon - \alpha\beta)}}$$

cuius propterea integralis est aequatio assumta.

23. Verum haec disparitas facile tollitur loco y ponend
ratio statim ex aequatione assumta potuisset esse manifest
via ad formulas dispares perveniendi, cuius hic exemplum
Assumatur aequatio

$$x^4 + 2axxyy + 2bxx = c,$$

cuius differentiale est

$$dx(x^3 + axyy + bx) + axxydy = 0$$

seu

$$\frac{dx}{xy} = \frac{-ady}{xx + ayy + b}.$$

Iam ex aequatione assumta primo determinetur xy per x

$$xy = \sqrt{\frac{c - 2bxx - x^4}{2a}},$$

tum vero $xx + ayy + b$ per y; at ob $(xx + ayy + b)^2 =$

$$xx + ayy + b = \sqrt{(c + (ayy + b)^2)}.$$

habebitur aequatio differentialis ista

$$\frac{dx\sqrt{2a}}{\sqrt{(e-2bxx-x^4)}} = \frac{ady}{\sqrt{(e+bb+2abyy+aay^4)}},$$

pleta integralis est assumta seu $y = \frac{\sqrt{(e-2bxx-x^4)}}{x\sqrt{2a}}$.

Etsi hoc integrale non est completum, tamen ex superioribus facile m redditur. Ponatur enim

$$\frac{ady}{\sqrt{(e+bb+2abyy+aay^4)}} = \frac{adz}{\sqrt{(e+bb+2abzz+aaz^4)}};$$

$+ bb$, $\gamma = 2ab$, $\zeta = aa$ erit

$$= \frac{z\sqrt{(e+bb)(e+bb+2abcc+aac^4)} + c\sqrt{(e+bb)(e+bb+2abzz+aaz^4)}}{e+bb-aacczz};$$

valor aequalis statuatur ipsi $\frac{\sqrt{(e-2bxx-x^4)}}{x\sqrt{2a}}$ et aequatio hinc inter x llana integralis erit completa huius aequationis differentialis

$$\frac{dx\sqrt{2a}}{\sqrt{(e-2bxx-x^4)}} = \frac{adz}{\sqrt{(e+bb+2abzz+aaz^4)}}.$$

am ex allatis patet, si haec bina membra insuper per numeros quaecunque multiplicentur, quemadmodum integrale completum inveniri oporteat.

Verum missa membrorum disparitate formationem parium membrorum s concipiamus; ponatur ergo

$$0 = \alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy + 2\varepsilon xy(x+y) + \zeta xxyy,$$

ferentiando obtinetur

$$x + \delta y + 2\varepsilon xy + \varepsilon yy + \zeta xyy) + dy(\beta+\gamma y+\delta x+2\varepsilon xy+\varepsilon xx+\zeta xxy) = 0$$

$$\frac{dy}{\beta+\gamma x+\delta y+2\varepsilon xy+\varepsilon yy+\zeta xyy} = \frac{-dx}{\beta+\gamma y+\delta x+2\varepsilon xy+\varepsilon xx+\zeta xxy}.$$

Ex resolutione autem aequationis assumtae elicitur

$$y = \frac{-\beta - \delta x - \varepsilon xx \pm \sqrt{(\beta\beta - \alpha\gamma + 2(\beta\delta - \alpha\varepsilon - \beta\gamma)x + (\delta\delta - \gamma\gamma - \alpha\zeta - 2\beta\varepsilon)xx + 2(\delta\varepsilon - \beta\zeta - \cdots}}{\gamma + 2\varepsilon x + \zeta xx}$$

Ponatur brevitatis gratia

$$\beta\beta - \alpha\gamma = A, \quad \beta\delta - \alpha\varepsilon - \beta\gamma = B, \quad \delta\delta - \gamma\gamma - \alpha\zeta - 2\ldots$$
$$\varepsilon\varepsilon - \gamma\zeta = E, \quad \delta\varepsilon - \beta\zeta - \gamma\varepsilon = D,$$

eritque

$$\beta + \delta x + \varepsilon xx + \gamma y + 2\varepsilon xy + \zeta xxy = \pm \sqrt{(A + 2Bx + Cxx + \ldots}$$
$$\beta + \delta y + \varepsilon yy + \gamma x + 2\varepsilon xy + \zeta xyy = \mp \sqrt{(A + 2By + Cyy + \ldots}$$

26. Hinc itaque concludimus huius aequationis differential…

$$\frac{dx}{\sqrt{(A + 2Bx + Cxx + 2Dx^3 + Ex^4)}} = \frac{dy}{\sqrt{(A + 2By + Cyy + 2Dy^3 \ldots}}$$

aequationem integralem eamque completam esse

$$0 = \alpha + 2\beta(x + y) + \gamma(xx + yy) + 2\delta xy + 2\varepsilon xy(x + y) + \ldots$$

adhibita scilicet superiori horum coefficientium determinati… autem definiatur β vel ε ex hac aequatione

$$\frac{BB(\varepsilon\varepsilon - E) - DD(\beta\beta - A)}{A\varepsilon\varepsilon - E\beta\beta} + \frac{2AD\varepsilon - 2BE\beta}{B\varepsilon - D\beta} = C;$$

tum vero erit

$$\gamma = \frac{A\varepsilon\varepsilon - E\beta\beta}{B\varepsilon - D\beta}, \quad \alpha = \frac{\beta\beta - A}{\gamma}, \quad \zeta = \frac{\varepsilon\varepsilon - E}{\gamma}$$

et

$$\delta = \frac{B\beta(\varepsilon\varepsilon - E) - D\varepsilon(\beta\beta - A)}{A\varepsilon\varepsilon - E\beta\beta} + \gamma \quad \text{seu} \quad \delta = \gamma + \frac{B + \ldots}{\beta \ldots}$$

27. Hinc ergo perspicuum est etiam hanc aequationem di…

$$\frac{dx}{\sqrt{(A + 2Dx^3)}} = \frac{dy}{\sqrt{(A + 2Dy^3)}}$$

posse; nam ob $B=0$, $C=0$ et $E=0$ erit

$$-\frac{DD(\beta\beta-A)}{A\varepsilon\varepsilon}-\frac{2A\varepsilon}{\beta}=0 \quad \text{seu} \quad \varepsilon=\sqrt[3]{\frac{DD}{2AA}\beta(A-\beta\beta)},$$

valores nimis prodeunt complicati. Facilius negotium absolvetur o valores litterarum evanescentium B, C et E; nam

$$E=0 \ \text{dat} \ \zeta=\frac{\varepsilon\varepsilon}{\gamma}; \quad \text{tum} \quad B=0 \ \text{dat} \ \delta=\gamma+\frac{\alpha\varepsilon}{\beta}$$

$$C=0 \ \text{dat} \ \delta\delta-\gamma\gamma=\alpha\zeta+2\beta\varepsilon=\frac{\alpha\varepsilon\varepsilon}{\gamma}+2\beta\varepsilon=\frac{\alpha^2\varepsilon\varepsilon}{\beta\beta}+\frac{2\alpha\gamma\varepsilon}{\beta},$$

tores sunt $\beta\beta=\alpha\gamma$ et $\alpha\varepsilon\varepsilon+2\beta\gamma\varepsilon=0$. At si esset $\beta\beta=\alpha\gamma$, foret in autem esset $\varepsilon=0$, foret et $\zeta=0$ et $D=0$, contra scopum. o oportet $\alpha\varepsilon=-2\beta\gamma$; unde fiet

$$\alpha=-\frac{2\beta\gamma}{\varepsilon}, \quad \delta=-\gamma \quad \text{et} \quad \zeta=\frac{\varepsilon\varepsilon}{\gamma}.$$

fieri debet

$$\beta\beta+\frac{2\beta\gamma\gamma}{\varepsilon}=A \quad \text{et} \quad -2\gamma\varepsilon-\frac{\beta\varepsilon\varepsilon}{\gamma}=D.$$

$=\frac{2\beta\gamma\gamma}{A-\beta\beta}$ et ob $\frac{\gamma D}{\varepsilon}=-(2\gamma\gamma+\beta\varepsilon)$ et $2\gamma\gamma+\beta\varepsilon=\frac{A\varepsilon}{\beta}$ erit $\frac{\gamma D}{\varepsilon}=-\frac{A\varepsilon}{\beta}$ $\varepsilon\varepsilon=-\frac{\beta\gamma D}{A}$. Ergo

$$\frac{4\beta\gamma^3}{(A-\beta\beta)^2}+\frac{D}{A}=0.$$

Cum autem tantum ratio litterarum A et D in censum veniat, aequatio lori absoluto ipsius A inveniendo inservit, quem autem nosse non Manebunt ergo litterae γ et β indeterminatae. Ponatur ergo

$$\gamma=-Ac \quad \text{et} \quad \beta=Dc;$$

$DDcc$ seu

$$\varepsilon=Dc \quad \text{hincque} \quad \delta=Ac, \quad \zeta=-\frac{DDc}{A} \quad \text{et} \quad \alpha=2Ac.$$

ius aequationis differentialis

$$\frac{dx}{\sqrt{(A+2Dx^3)}}=\frac{dy}{\sqrt{(A+2Dy^3)}}$$

integrale est

$$0 = 2A + 2D(x+y) - A(xx+yy) + 2Axy + 2Dxy(x+y) - \frac{D}{A}$$

Hoc autem integrale non est completum, tale autem redditu $\gamma = -A$ et $\beta = Dcc$, unde fit $\varepsilon\varepsilon = DDcc$ et $\varepsilon = Dc$; porro e $\zeta = -\frac{DDcc}{A}$, $\alpha = 2Ac$, ita ut integrale completum sit

$$0 = 2Ac + 2Dcc(x+y) - A(xx+yy) + 2Axy + 2Dcxy(x+y) -$$

ubi c est constans ab arbitrio pendens; unde fit

$$y = \frac{Dcc + Ax + Dcxx \pm \sqrt{c\left(2A + \frac{DD}{A}c^3\right)(A + 2Dx^3)}}{A - 2Dcx + \frac{DDcc}{A}xx}$$

29. Hic casus notari meretur, quo $A = 1$ et $D = \frac{1}{2}$, ut hab aequatio differentialis

$$\frac{dx}{\sqrt{(1+x^3)}} = \frac{dy}{\sqrt{(1+y^3)}},$$

ubi ad fractiones tollendas loco c scribatur $2c$, eritque integrale

$$0 = 4c + 4cc(x+y) - xx - yy + 2xy + 2cxy(x+y) - ccx$$

seu

$$y = \frac{2cc + x + cxx \pm 2\sqrt{c(1+c^3)(1+x^3)}}{1 - 2cx + ccxx}.$$

Integralia ergo particularia erunt

$$\text{I. si } c = 0, \quad y = x;$$

$$\text{II. si } c = \infty, \quad y = \frac{2 \pm 2\sqrt{(1+x^3)}}{xx};$$

$$\text{III. si } c = -1, \quad y = \frac{2 + x - xx}{1 + 2x + xx} = \frac{2-x}{1+x}.$$

30. Ex eodem principio, si in § 26 loco litterarum A, B, C, I per quantitatem quampiam p multiplicentur, nihilo minus aequa

ita erit

$$\frac{dx}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}} = \frac{dy}{\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}}$$

ponaturque

$$p = \frac{BB\epsilon\epsilon - DD\beta\beta}{BBE - ADD} + 2\frac{(AD\epsilon - BE\beta)(A\epsilon\epsilon - E\beta\beta)}{(B\epsilon - D\beta)(BBE - ADD)} - \frac{C(A\epsilon\epsilon - E\beta\beta)}{BBE - ADD},$$

erit

$$\ldots = \frac{A\epsilon\epsilon - E\beta\beta}{B\epsilon - D\beta}, \quad \alpha = \frac{\beta\beta - Ap}{\gamma}, \quad \zeta = \frac{\epsilon\epsilon - Ep}{\gamma} \quad \text{atque} \quad \delta = -\gamma + \frac{\alpha\epsilon + \ldots}{\ldots}$$

ut litterae β et ϵ maneant indeterminatae, fietque propterea a-
egralis completa

$$0 = \alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy + 2\epsilon xy(x+y) + \zeta xxyy$$

de fit

$$y = \frac{-\beta - \delta x - \epsilon xx + \sqrt{p(A+2Bx+Cxx+2Dx^3+Ex^4)}}{\gamma + 2\epsilon x + \zeta xx}.$$

31. Notandum denique est non solum hanc aequationem differen-
us integrale completum modo exhiberi, sed etiam hanc multo latius pa-

$$\frac{m\,dx}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}} = \frac{n\,dy}{\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}}$$

per algebraice et quidem complete integrari posse, dummodo coe-
n m et n ratio fuerit rationalis; haec enim integratio simili modo
ur, quo supra usus sum ad aequationem, quae mihi hic praecipu-
posita, integrandam. Methodus autem, cuius hic specimina attu-
hi videtur comparata, ut indolem eius diligentius excolendo ad i-
s apta reddi queat, unde haud contemnenda commoda in Analys-
undatura.

32. Hic autem observo formulam § 26 assumtam latius extendend-
di differentialia inter se comparari posse, quae sint disparia, atqu-

exemplum disparitatis (§ 22) allatum hoc modo obtineri posse, it
quae hactenus sunt tradita, in hac generali investigatione c
Fingatur scilicet haec aequatio integralis

$$(1)\quad \alpha xxyy + 2\beta xxy + 2\gamma xyy + \delta xx + \varepsilon yy + 2\zeta xy + 2\eta x + 2\theta y$$

ex qua fit

$$(2)\quad y = \frac{-\beta xx - \zeta x - \theta + \sqrt{((\beta xx + \zeta x + \theta)^2 - (\alpha xx + 2\gamma x + \varepsilon)(\delta xx + 2}{\alpha xx + 2\gamma x + \varepsilon}$$

$$(3)\quad x = \frac{-\gamma yy - \zeta y - \eta - \sqrt{((\gamma yy + \zeta y + \eta)^2 - (\alpha yy + 2\beta y + \delta)(\varepsilon yy + 2}{\alpha yy + 2\beta y + \delta}$$

Ponatur iam brevitatis gratia

$App = \beta\beta - \alpha\delta$	$\mathfrak{A}qq = \gamma\gamma - \alpha\varepsilon$
$2Bpp = 2\beta\zeta - 2\alpha\eta - 2\gamma\delta$	$2\mathfrak{B}qq = 2\gamma\zeta - 2\alpha\theta - 2\beta$
$Cpp = \zeta\zeta + 2\beta\theta - \alpha\varkappa - \delta\varepsilon - 4\gamma\eta$	$\mathfrak{C}qq = \zeta\zeta + 2\gamma\eta - \alpha\varkappa -$
$2Dpp = 2\zeta\theta - 2\gamma\varkappa - 2\varepsilon\eta$	$2\mathfrak{D}qq = 2\zeta\eta - 2\beta\varkappa - 2\delta$
$Epp = \theta\theta - \varepsilon\varkappa$	$\mathfrak{E}qq = \eta\eta - \delta\varkappa$

eritque

$$(4)\quad p\sqrt{(Ax^4 + 2Bx^3 + Cxx + 2Dx + E)} = \alpha xxy + 2\gamma xy + \varepsilon y + \beta x$$

$$(5)\quad -q\sqrt{(\mathfrak{A}y^4 + 2\mathfrak{B}y^3 + \mathfrak{C}yy + 2\mathfrak{D}y + \mathfrak{E})} = \alpha xyy + 2\beta xy + \delta x + \gamma y$$

33. At si aequatio integralis assumta differentietur, fiet

$$(6)\quad \begin{aligned} &dx(\alpha xyy + 2\beta xy + \gamma yy + \delta x + \zeta y + \eta) \\ &+ dy(\alpha xxy + \beta xx + 2\gamma xy + \varepsilon y + \zeta x + \theta) = 0, \end{aligned}$$

unde, si istorum factorum valores (4) et (5) reperti substituan
ista aequatio differentialis

$$(7)\quad \frac{q\,dx}{\sqrt{(Ax^4 + 2Bx^3 + Cxx + 2Dx + E)}} = \frac{p\,dy}{\sqrt{(\mathfrak{A}y^4 + 2\mathfrak{B}y^3 + \mathfrak{C}yy + 2\mathfrak{D}}}$$

cuius proptorea integralis est aequatio assumta (1).

Cum autem supra habeantur 10 aequationes, coefficientium autem α ... etc. numerus sit 9, quorum unus pro lubitu assumi potest, octo re...unt litterae determinandae. Porro autem insuper definiendae accede... litterae p et q, ita ut nunc decem quantitates adsint incognitae, ... coefficientes utriusque formulae A, B, C, D, E et $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$ ...ur pro lubitu assumi posse. Verum perspicuum est, cum alteri ...int ad libitum assumti, alteros non omnino ab arbitrio nostro pend... ...s enim quaevis formula ad algebraicam reduci posset.

34. Hinc autem aliae datae formulae transmutationes non inelegan... ...neri possunt, si loco y alii valores substituantur. Veluti si pona... $= 0$ seu $\eta\eta = \delta x$ statuaturque $y = zz$, sequens prodibit aequa... ...rentialis

$$(8)\qquad \frac{q\,dx}{\sqrt{(Ax^4 + 2Bx^3 + Cxx + 2Dx + E)}} = \frac{2p\,dz}{\sqrt{(\mathfrak{A}z^8 + 2\mathfrak{B}z^4 + \mathfrak{C}z^2 + 2\mathfrak{D})}},$$

...s propterea integralis est aequatio assumta, si ponatur $y = zz$ statua... $\eta\eta = \delta x$ ac reliquae litterae rite determinentur. Integrale etiam c... ...um nulla difficultate reperietur; nam etiamsi fortasse integrale invent... ...um non involvat constantem, ponatur

$$\frac{q\,dx}{\sqrt{(Ax^4 + 2Bx^3 + Cxx + 2Dx + E)}} = \frac{q\,du}{\sqrt{(Au^4 + 2Bu^3 + Cuu + 2Du + E)}}$$

...uius aequationis integrale completum ex antecedentibus assignare lic... ...e hinc integrale quoque completum aequationis ex formulis dispari... ...tantis colligetur.

35. Quemadmodum huius aequationis differentialis, ut a simplicissi... ...piam,

$$\frac{dx}{\sqrt{(f + gx)}} = \frac{dy}{\sqrt{(f + gy)}}$$

...grale completum est

$$gg(xx + yy) - 2ggxy - 2ccg(x + y) + c^4 - 4ccf = 0,$$

deinde vero huius aequationis differentialis

$$\frac{dx}{\sqrt{(f+gxx)}}=\frac{dy}{\sqrt{(f+gyy)}}$$

integrale completum est

$$xx+yy-2xy\sqrt{(1+fgcc)}-ccff=0,$$

tertio vero huius aequationis differentialis

$$\frac{dx}{\sqrt{(f+gx^3)}}=\frac{dy}{\sqrt{(f+gy^3)}}$$

integrale completum est

$$f(xx+yy)+\frac{ggcc}{4f}xxyy-gcxy(x+y)-2fxy-gcc(x+$$

quarto porro huius aequationis differentialis

$$\frac{dx}{\sqrt{(f+gx^4)}}=\frac{dy}{\sqrt{(f+gy^4)}}$$

integrale completum repertum est

$$f(xx+yy)-fcc-gccxxyy-2xy\sqrt{f(f+gc^4)}$$

ita etiam integrale completum huius aequationis

$$\frac{dx}{\sqrt{(f+gx^6)}}=\frac{dy}{\sqrt{(f+gy^6)}}$$

reperiri poterit.

36. Determinentur primo in § 33 valores, ita ut pro

$$\frac{dx}{\sqrt{(fx+gx^4)}}=\frac{dy}{\sqrt{(fy+gy^4)}},$$

cuius integralis completa reperitur

$$gg(xx+yy)-4ggcxxyy-4fgccxy(x+y)-2ggxy-2fgc$$

ac $x = tt$ et $y = uu$, ut prodeat haec aequatio differentialis

$$\frac{dt}{\sqrt{(f + gt^6)}} = \frac{du}{\sqrt{(f + gu^6)}},$$

erea integralis completa erit

$$4ggct^4u^4 - 4fgcttuu(tt + uu) - 2ggttuu - 2fgc(tt + uu) + ffcc = 0;$$

meretur casus ex hypothesi $c = \infty$ resultans, qui dat

$$4gttuu(tt + uu) = f.$$

OBSERVATIONES DE COMPARATION
CURVARUM IRRECTIFICABIL

Commentatio 252 indicis ENESTROEMIANI
Novi commentarii academiae scientiarum Petropolitanae 6 (1756/7)
Summarium ibidem p. 10—11

SUMMARIUM

Haec dissertatio ex eodem fonte est petita atque antecedens.
methodo formulas integrales, quae neque algebraice neque per a
expediri queant, algebraice inter se comparandi. Methodus autem
negotium conficitur, ita est comparata, ut non data opera sit inve
quasi detecta; ex quo, cum ad inventiones alias abstractissimas per
videtur, ut omni studio uberius excolatur. In superiori quidem di
praestitum, ut omnium curvarum, quarum arcus indefinite huius
$\int \frac{adz}{\sqrt{(A + Bz + Cz^2 + Dz^3 + Ez^4)}}$ exprimuntur, arcus quicunque inter
arcu quovis alii arcus ad eum datam rationem tenentes geometrice
omnino modo, quo arcus circulares inter se comparari solent. Tali a
curva lemniscata vocari solita, cuius arcus indefinite hac formula
huiusque arcuum comparatio in hac dissertatione prolixius explicatu
Auctor investigationes suas ad arcus ellipticos et hyperbolicos
omnino vis illius methodi cernitur, cum rectificatio ellipsis et hy
formulam integralem ante commemoratam revocari possit. Neque v
comparatio arcuum uti in circulo institui potest; sed, quod iam p
bolicis est factum, id nunc etiam istius novae methodi beneficio
praestatur. Scilicet dato in altera curva arcu quocunque a puncto
arcus in eadem curva abscindi potest, cuius ab illo differentiam geo
tum vero etiam negotium ita confici potest, ut non ipsorum arcuum

quarum differentia sit geometrice assignabilis, idque ita, ut arcus quaesitus i
to incipiat. Omissa autem hac conditione, ut arcus quaesitus in dato puncto
r, effici potest, ut differentia vel ipsorum arcuum vel quorundam multiplorum cor
eneat sicque arcus assignari queant, qui *absolute datam inter se teneant rat*
e hinc istud problema maxime notatu dignum resolvi potest, quo datus quic
s, sive ellipticus sive hyperbolicus, ita secari iubetur, ut partium differentia geom
nabilis evadat. Sub finem animadvertit Auctor, quam insignia incrementa in A
torum hinc expectari queant, cum inde eiusmodi aequationum differentialium,
alii methodo cedant, integralia adeo algebraica assignari possint.

Speculationes mathematicae, si ad earum utilitatem respicimus, ad
ses reduci debere videntur; ad priorem referendae sunt eae, quae cu
m commodum tum ad alias artes insigne aliquod commodum affe
rum propterea pretium ex magnitudine huius commodi statui
ra autem classis eas complectitur speculationes, quae, etsi cum nulli
i commodo sunt coniunctae, tamen ita sunt comparatae, ut ad fines
ea promovendos viresque ingenii nostri *acuendas occasionem praeb*
n enim plurimas investigationes, unde maxima utilitas expectari p
utum analyseos defectum deserere cogimur, non minus pretium iis s
uibus statuendum videtur, quae haud contemnenda Analyseos increm
rentur. Ad hunc autem scopum inprimis accommodatae videntur
i *observationes*, quae *cum quasi casu sint factae et a posteriori det*
e ad easdem a priori ac per viam directam perveniendi minus vel
am est perspecta. Sic enim cognita iam veritate facilius in eas meth
inquirere licebit, quae ad eam directe sint perducturae, novis autem meth
estigandis Analyseos fines non mediocriter promoveri nullum plane
ium.

Huiusmodi autem observationes, quae nulla certa methodo sunt f
rumque ratio non parum abscondita videtur, nonnullas deprehen
o Ill. Comitis FAGNANI[1]) nuper in lucem edito; quae idcirco omni a
e dignae sunt censendae neque studium, quod in ulteriori earum in
one consumitur, inutiliter collocatum erit iudicandum. Commemorar
am in hoc libro quasdam eximias proprietates, quibus curvae *El*

1) G. C. FAGNANO, *Produzioni matematiche*; vide notam p. 59. A. K.

perbola et *Lemniscata* sunt praeditae, harumque curvarum arcus di
er se comparantur; cum igitur ratio harum proprietatum maxime oc
eatur, haud alienum fore arbitror, si eas diligentius examinavero, et
hi insuper circa has curvas elicere contigit, cum publico communica

Quod igitur primum ad has curvas attinet, notum est earum rect
nem omnes Analyseos vires transcendere, ita ut earum arcus non so
n algebraice exprimi, sed etiam neque idem ad quadraturam circuli
perbolae reduci queant. Quare eo magis mirum videri debet, quod
nnes FAGNANO invenit, Ellipsi et Hyperbola infinitis modis eiusmodi b
cus exhiberi posse, quorum differentia geometrice assignari queat, in c
nniscata autem infinitis modis eiusmodi dari arcus binos, qui inter se
t aequales vel alter ad alterum rationem duplam teneat, unde dein
odum colligit in hac curva etiam eiusmodi arcus assignandi, qui aliam
rationem teneant.

Pro Ellipsi quidem et Hyperbola nihil admodum mihi praeterea scr
uit; unde contentus ero faciliorem constructionem eorum arcuum dec
orum differentia geometrice exhiberi queat. Pro curva autem lemnis
dem vestigiis insistens multo plures, imo infinitas elicui formulas, qu
neficio non solum infinitis modis eiusmodi binos arcus definire possum,
ter se vel sint aequales vel rationem teneant duplam, sed etiam qui
ter se in ratione quacunque numeri ad numerum.

I. DE ELLIPSI

1. Sit quadrans ellipticus ABC (Fig. 1), cuius centrum in C, eiu
miaxes ponantur $CA = 1$ et $CB = c$; sumta ergo abscissa quacunque CP

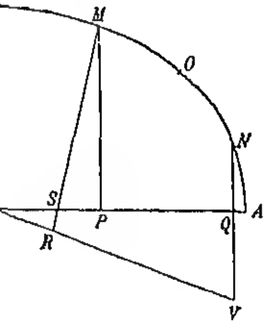


Fig. 1.

erit applicata ei respondens $PM = y = c\sqrt{(1 -$
cuius differentiale cum sit $dy = -\frac{cx\,dx}{\sqrt{(1-xx)}}$,
abscissae $CP = x$ arcus ellipticus respondens

$$BM = \int \frac{dx\sqrt{(1-(1-cc)xx)}}{\sqrt{(1-xx)}}.$$

Ponatur brevitatis gratia $1 - cc = n$, ut sit ar

$$BM = \int dx \sqrt{\frac{1-nxx}{1-xx}},$$

cui abscissa $CQ = u$ erit simili modo arcus ei respondens

$$BN = \int du \sqrt{\frac{1-nuu}{1-uu}}.$$

quaeritur, quomodo hae duae abscissae x et u inter se comparatae
nt, ut arcuum summa

$$BM + BN = \int dx \sqrt{\frac{1-nxx}{1-xx}} + \int du \sqrt{\frac{1-nuu}{1-uu}}$$

evadat seu geometrice exhiberi queat.

uestio ergo huc redit, ut determinetur, cuiusmodi functio ipsius x
alitati debeat, ut formula differentialis

$$dx \sqrt{\frac{1-nxx}{1-xx}} + du \sqrt{\frac{1-nuu}{1-uu}}$$

em admittat. Facile autem perspicitur, si haec quaestio in genere
r, eius solutionem utriusque formulae integrationem inniti ideoque
lyseos fines transgredi atque ipsam ellipseos rectificationem. Cum
tio generalis nullo modo exspectari queat, in solutiones particulares
endum, quae uti nulla certa ratione reperiri possunt, ita etiam
casui et coniecturae erit tribuendum; ex quo earum verum funda-
tiamsi ipsae sint cognitae[, vix poterit cognosci].

imum quidem statim occurrit casus $u = -x$, quo formula nostra
is in nihilum abit; sed quia hinc duo Ellipsoos arcus aequales et
untur, uti hic casus nimis est obvius, ita etiam quaestioni propo-
ne satisfacere est censendus. Cum igitur tentaminibus totum nego-
vi debeat, fingatur

$$\sqrt{\frac{1-nxx}{1-xx}} = \alpha u$$

ncipiatur, ut vicissim fiat

$$\sqrt{\frac{1-nuu}{1-uu}} = \alpha x;$$

abebitur

$$BM + BN = \alpha \int u\,dx + \alpha \int x\,du = \alpha xu + \text{Const.},$$

$$1 - nxx - aauu + aauuxx = 0 \quad \text{quam} \quad 1 - nuu - aaxx + aa$$

unde patet statui debere $aa = n$ et $a = \sqrt{n}$, ita ut

$$u = \sqrt{\frac{1 - nxx}{n - nxx}} \quad \text{et} \quad BM + BN = xu\sqrt{n} + \text{Const.}$$

4. Etsi autem hoc modo quaestioni satisfactum videtur, determinationes in Ellipsi locum habere nequeunt. Nam cum sit $n = 1 - cc$, erit $n - nxx < 1 - nxx$ ideoque $u > 1$; abscissa org axem CA superaret eique propterea arcus imaginarius respond hinc nulla conclusio conformis deduci posset.

5. Tentemus ergo alias formulas sitque tam

$$\sqrt{\frac{1 - nxx}{1 - xx}} = \frac{a}{u} \quad \text{quam} \quad \sqrt{\frac{1 - nuu}{1 - uu}} = \frac{a}{x},$$

unde ob

$$aa - aaxx - uu + nxxuu = 0 \quad \text{et} \quad aa - aauu - xx + nxx$$

colligimus $a = 1$, ita ut sit

$$1 - uu - xx + nxxuu = 0 \quad \text{ideoque} \quad u = \sqrt{\frac{1 - xx}{1 - nxx}}.$$

Hinc autem prodit

$$BM + BN = \int \frac{dx}{u} + \int \frac{du}{x} = \int \frac{xdx + udu}{xu}.$$

Verum aequatio $uu + xx = 1 + nxxuu$ differentiata dat

$$xdx + udu = nxu(xdu + udx) \quad \text{seu} \quad \frac{xdx + udu}{xu} = n(xdu +$$

unde concludimus

$$BM + BN = n\int(xdu + udx) = nxu + \text{Const.}$$

6. Haec solutio nullo incommodo laborat; cum enim sit $1 - nxx > 1 - xx$ ideoque $u < 1$, uti natura rei postulat. Su

quacunque $CP = x$ capiatur altera

$$CQ = u = \sqrt{\frac{1 - xx}{1 - nxx}}$$

ue summa arcuum $BM + BN = nxu + \text{Const.}$ Ad quam consta
iendam sit $x = 0$, ut fiat $BM = 0$; eritque $u = 1$ et arcus BN a
rantem $BMNA$; unde fit $0 + BMNA = 0 + \text{Const.}$ sicque haec co
$= BMNA$. Quo valore eius loco substituto habemus

$$BM + BN = nxu + BMNA$$

que

$$BM - AN = nxu = (1 - cc)xu = BN - AM.$$

7. Dato ergo in quadrante elliptico ACB puncto quocunque M ass
nus alterum punctum N, ita ut differentia arcuum $BM - AN$, vel
est aequalis $BN - AM$, geometrice exprimi queat. Quod quo f
stari possit, ducamus ad Ellipsin in puncto M normalem MS; erit
alis $PS = ccx$ et ob $PM = c\sqrt{(1 - xx)}$ ipsa normalis

$$MS = c\sqrt{(1 - xx + ccxx)} = c\sqrt{(1 - nxx)};$$

quo pro altero puncto N abscissa erit $CQ = u = \frac{PM}{MS}CA$. Vel in
m MS productam ex C demittatur perpendicularis CR, quae prod
, ut sit $CV = CA = 1$, et ob $\frac{CR}{CS} = \frac{PM}{MS}$ erit $CQ = \frac{CR}{CS}CV$. Qu
to V in axem CA ducatur perpendicularis VQ, quae punctum Q e
a ipsum punctum N designabit.

8. Cum sit $PS = ccx$, erit $CS = x - ccx = nx$ ideoque

$$CR = \frac{CQ \cdot CS}{CV} = \frac{u \cdot nx}{1} = nux.$$

ergo ipsum perpendiculum CR differentiam arcuum $BM - A$
$- AM$ exhibebit. Arcuum ergo hoc modo designatorum differenti
$x\sqrt{\frac{1 - xx}{1 - nxx}}$, quae igitur evanescit tam casu $x = 0$ quam $x = 1$,
cta M et N in ipsa puncta B et A incidunt. Maxima autem

differentia evadit, si $nx^4 - 2xx + 1 = 0$, hoc est si $x =$
$x = u$ et ambo puncta M et N in unum punctum O
casu differentia arcuum $BO - AO = nxx = 1 - c$ ideo
differentiae $CA - CB$ fiet aequalis, ita ut sit $CA + AO$

9. Si punctum M in ipso hoc puncto O capiatur,

$$CP = x = \frac{1}{\sqrt{(1+c)}},$$

erit

$$PM = \frac{c\sqrt{c}}{\sqrt{(1+c)}} \quad \text{et} \quad PS = \frac{cc}{\sqrt{(1+c)}}$$

hincque $MS = c\sqrt{c}$, unde variis modis situs puncti
poterit. Cum autem sit

$$CM = CO = \frac{\sqrt{(1+c^3)}}{\sqrt{(1+c)}} = \sqrt{(1 - c + cc)} = \sqrt{(1 + cc}$$

unde facilis constructio deducitur, sequentia ergo Th
visum est, quorum demonstratio ex allatis est manifest

THEOREMA 1

10. *In quadrante elliptico ACB* (Fig. 2) *si ad puncti*
tangens HMK, quae cum altero axe CB in H concurra
CA aequalis capiatur, ut sit
per K axi CB parallela aga
in N, arcuum BM et AN
geometrice assignari poterit;
C in tangentem perpendiculo
differentia $BM - AN = M$

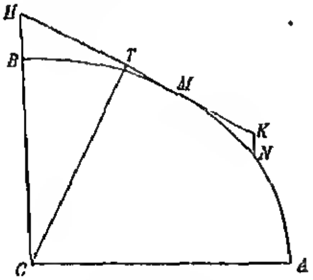


Fig. 2.

Demonstratio ex figu
tangens HMK sit rectae i
parallela et aequalis; tum vero perspicuum est esse M

THEOREMA 2

Si super quadrantis elliptici ACB (Fig. 3) *altero semiaxe CA triangulum rum CAE constituatur et in eius latere AE portio capiatur AF = CB ue CF aequalis applicetur in ellipsi recta CO, O hanc habebit proprietatem, ut sit*

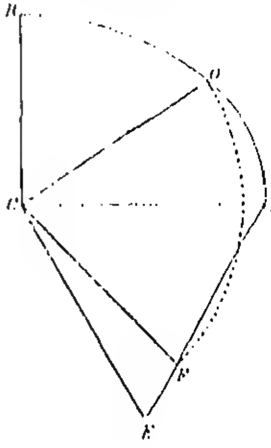


Fig. 3.

$$CA + \text{arcu } AO = CB + \text{arcu } BO.$$

monstratio ex § 9 evidens est. Cum enim sit

$$CA = 1, \quad AF = c \quad \text{et} \quad \text{ang. } CAF = 60^\circ,$$

$$CF = \sqrt{(1 + cc - 2c \cos. 60^\circ)}$$

$$= CO.$$

II. DE HYPERBOLA

Sit C (Fig. 4) centrum hyperbolae AMN eiusque semiaxis transversus , semiaxis coniugatus $= c$; erit sumta abscissa quacunque $CP =$ a $PM = c\sqrt{(xx - 1)}$ eiusque differentiale $= \frac{cxdx}{\sqrt{(xx-1)}}$; unde fit arcus

$$AM = \int \frac{dx\sqrt{((1 + cc)xx - 1)}}{\sqrt{(xx - 1)}}.$$

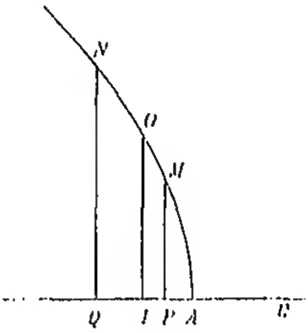


Fig. 4.

r brevitatis gratia $1 + cc = n$; erit

$$AM = \int dx \sqrt{\frac{nxx - 1}{xx - 1}}.$$

rgo modo si capiatur alia quaevis abscissa u, erit arcus ei respondens

$$AN = \int du \sqrt{\frac{nuu - 1}{uu - 1}}.$$

alterum N ita definiatur, ut summa arcuum $AM + AN$

$$\int dx\sqrt{\frac{nxx-1}{xx-1}} + \int du\sqrt{\frac{nuu-1}{uu-1}}$$

absolute integrationem admittat; quod quidem evenire ca
patet; verum hinc nihil ad institutum nostrum concludere

14. Ponamus ergo

$$\sqrt{\frac{nxx-1}{xx-1}} = u\sqrt{n},$$

cum hinc vicissim fiat

$$\sqrt{\frac{nuu-1}{uu-1}} = x\sqrt{n};$$

utrinque enim prodit haec aequatio $nuuxx - n(uu + xx) +$
hac hypothesi prodit summa arcuum

$$AM + AN = \int u\,dx\sqrt{n} + \int x\,du\sqrt{n} = ux\sqrt{n} +$$

Haec ergo integrabilitas ut locum habeat, oportet sit $u =$
ob $n > 1$ prodeat quoque $u > 1$, ex dato puncto M semper
assignari poterit.

15. Ad constantem definiendam patet casum $x = 1$,
verticem A incidit, nihil iuvare, cum inde oriatur $u = \infty$
infinitum removeatur. Quocirca ut haec constans debite
casum considerari oportet; potior autem non occurrit qua
et N in unum coalescunt seu quo fit $u = x$ et $nx^4 - 2$
autem oritur

$$xx = 1 + \frac{c}{\sqrt{(1+cc)}} \quad \text{et} \quad x = \sqrt{\left(1 + \frac{c}{\sqrt{(1+}\right.}$$

16. Sit igitur O hoc punctum, in quo ambo puncta
ductaque applicata OI erit abscissa

$$CI = \sqrt{\left(1 + \frac{c}{\sqrt{(1+cc)}}\right)} \quad \text{et} \quad 2AO = c + \sqrt{(1 +}$$

obtinemus constantem quaesitam

$$= 2AO - c - \sqrt{(1+cc)}$$

$(1+cc)$. Quo valore substituto erit pro quibusvis punctis M et N sumtis, ut sit $u = \sqrt{\frac{nxx-1}{nxx-n}}$, summa arcuum

$$AM + AN = ux\sqrt{n} + 2AO - c - \sqrt{(1+cc)}$$

$$ON - OM = ux\sqrt{n} - c - \sqrt{(1+cc)}.$$

duos arcus nacti sumus ON et OM, quorum differentia $ON - OM$ assignari potest.

o autem facilius pateat, quomodo tam punctum O quam ex puncto n N definiri queat, erigatur in A (Fig. 5) perpendiculum $AD = c$ cta CD hyperbolae asymtota; s $CP = x$, $PM = y$ ducatur T; erit ob

$$(xx-1) \quad \text{et} \quad dy = \frac{cxdx}{\sqrt{(xx-1)}}$$

$$\frac{xx-1)}{cx} = x - \frac{1}{x} \quad \text{et} \quad CT = \frac{1}{x}$$

ngens

$$MT = \frac{y\sqrt{(nxx-1)}}{cx}.$$

t

$$\frac{xx-1}{xx-1} = \frac{PT}{MT} \quad \text{ideoque} \quad u = \frac{MT}{PT\sqrt{(1+cc)}} = \frac{CA^2 \cdot MT}{CD \cdot PT} = CQ.$$

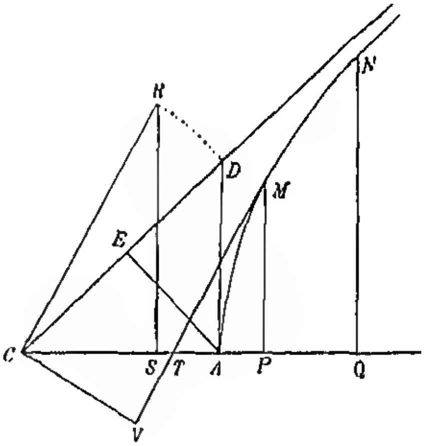


Fig. 5.

ucatur ex centro C tangenti TM parallela $CR = CD$ demissoque axem perpendiculo RS erit $CS = \frac{CD \cdot PT}{MT}$ ideoque $CQ = \frac{CA^2}{CS}$. capienda erit tertia proportionalis ad CS et CA. Commodius

cum sit

$$QN = \frac{cc}{\sqrt{n(xx-1)}} = \frac{c^3}{y\sqrt{n}},$$

erit

$$PM \cdot QN = \frac{c^3}{\sqrt{(1+cc)}} = \frac{AD^3}{CD}$$

vel demisso ex A in asymtotam perpendiculo AE erit

$$PM \cdot QN = AD \cdot DE$$

ob $DE = \frac{AD^2}{CD}$, unde sequens Theorema conficitur.

THEOREMA 3

19. *Existente AOZ* (Fig. 6) *hyperbola, C eius centro,*
asymtota, ad quam ex A axi perpendiculariter ducta sit
ad asymtotam perpendic
stituatur IO media pro
DE atque utrinque ap
statuantur, ut inter eas
tionalis, tum arcuum
geometrice assignari pot

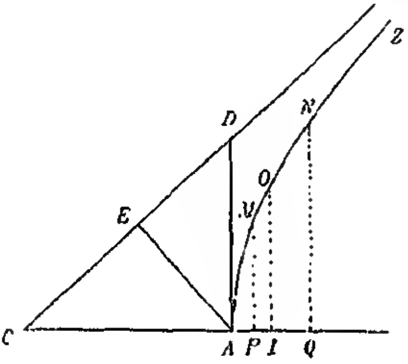


Fig. 6.

$$ON - OM = \frac{C}{\cdots}$$

Demonstratio ex
dente est manifesta.

et N in O coeuntibus sit $IO \cdot IO = AD \cdot DE$, erit IO
inter AD et DE; hacque inventa esse oportet $PM \cdot$
vero ex § 16 intelligitur esse $ON - OM = (CP \cdot CQ$
$\sqrt{n} = CD$ erit homogeneitatem implendo $ON - OM = ($
At est $\frac{CA^2}{CD} = CE$ sicque constat Theorematis veritas.

III. DE CURVA LEMNISCATA

20. Haec curva ob plurimas, quibus praedita est, insi
Geometras est celebrata, imprimis autem, quod eius

sunt aequales. Natura autem huius curvae ita est comparata, ut rdinatis orthogonalibus $CP = x$, $PM = y$ (Fig. 7) ista aequatione ex-

$$(xx + yy)^2 = xx - yy.$$

et hanc curvam esse lineam inis, quae in C, quod punctum m dicitur, cum axe CA angulum constituit, in A autem sumta xem normaliter traiicit. Figura MNA quartam partem totius exhibet, cui tres reliquae ca centrum C aequales sunt ae; id quod inde liquet, quod, ssa x sive applicata y sive utraque negativum valorem induat, ndem manet.

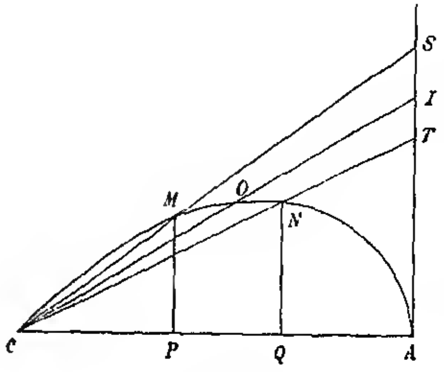


Fig. 7.

uod igitur ad expressionem arcus cuiusque CM huius curvae attinet, lissime ex corda CM definitur. Si enim hanc cordam ponamus ob $xx + yy = zz$ habebimus $z^4 = xx - yy = 2xx - zz = zz - 2yy$, ius

$$x = z\sqrt{\frac{1 + zz}{2}} \quad \text{et} \quad y = z\sqrt{\frac{1 - zz}{2}}$$

iando

$$dx = \frac{dz(1 + 2zz)}{\sqrt{2(1 + zz)}} \quad \text{et} \quad dy = \frac{dz(1 - 2zz)}{\sqrt{2(1 - zz)}}.$$

elementum arcus CM colligitur

$$\sqrt{(dx^2 + dy^2)} = dz\sqrt{\frac{(1 - zz)(1 + 2zz)^2 + (1 + zz)(1 - 2zz)^2}{2(1 + zz)(1 - zz)}}$$

$$\sqrt{(dx^2 + dy^2)} = \frac{dz}{\sqrt{(1 - z^4)}}.$$

ergo corda quaecunque ex centro C educta ponatur $CM = z$, erit a subtensus $CM = \int \frac{dz}{\sqrt{(1 - z^4)}}$. Simili ergo modo si alia quaevis dicatur $= u$, erit arcus ab ea subtensus $CN = \int \frac{du}{\sqrt{(1 - u^4)}}$, cuius

docuit, cuiusmodi functio ipsius z capi debeat pro u, ut vel
fiat arcui CM, vel ut arcus CN sit duplus arcus CM,
AN sit aequalis duplo arcui CM. Hos ergo casus primo
autem, quae mihi circa alias huiusmodi arcuum proportio
in medium sum allaturus.

THEOREMA 4

23. *In curva lemniscata hactenus descripta si applicet*
$CM = z$ aliaque insuper applicetur, quae sit

$$CN = u = \sqrt{\frac{1-zz}{1+zz}},$$

erit arcus CM aequalis arcui AN vel etiam arcus CN aequ

DEMONSTRATIO

Cum sit corda $CM = z$, erit arcus $CM = \int \frac{dz}{\sqrt{(1-z^4)}}$ et
erit arcus $CN = \int \frac{du}{\sqrt{(1-u^4)}}$. At est $u = \sqrt{\frac{1-zz}{1+zz}}$; unde fit

$$du = \frac{-2z\,dz}{(1+zz)\sqrt{(1-z^4)}}.$$

Praeterea vero est

$$u^4 = \frac{1-2zz+z^4}{1+2zz+z^4} \quad \text{ideoque} \quad 1-u^4 = \frac{4zz}{(1+zz)^2} \quad \text{et} \quad \sqrt{(1$$

Quibus valoribus substitutis habebitur

$$\text{arc. } CN = -\int \frac{dz}{\sqrt{(1-z^4)}} = -\text{arc. } CM + \text{Co}$$

arc. CN + arc. CM = Const. Ad hanc cons
casus, quo $z = 0$ ideoque et arcus $CM = 0$;
$u = 1 = CA$ ideoque arcus CN abit in quad
r pro hoc casu $CMNA + 0$ = Const. Hoc e

..ibit in genere arc. CN + arc. CM = arc. $CMNA$ hincque

$$\text{arc. } CM = \text{arc. } AN$$

..rcum MN utrinque addendo

$$\text{arc. } CMN = \text{arc. } ANM.$$

..E. D.

COROLLARIUM 1

24. Dato ergo quocunque arcu CM in centro C terminato, cuius c.. $CM = z$, si ab altera parte seu vertice A abscindetur arcus aequalis ..ondo cordam

$$CN = u = \sqrt{\frac{1-zz}{1+zz}} \quad \text{seu} \quad CN = CA\sqrt{\frac{CA^2 - CM^2}{CA^2 + CM^2}}$$

..ogeneitatem supplendo per axem $CA = 1$.

COROLLARIUM 2

25. Cum sit $u = \sqrt{\frac{1-zz}{1+zz}}$, erit vicissim $z = \sqrt{\frac{1-uu}{1+uu}}$; unde cordas CM et.. ..r se permutare licet, ita ut, si ambae cordae $CM = z$ et $CN = u$.. ..int comparatae, ut sit

$$uuzz + uu + zz = 1,$$

..u puncta M et N inter se permutari queant indeque prodeat .. CM = arc. AN quam arc. CN = arc. AM.

COROLLARIUM 3

26. Cum sit $CN = u = \sqrt{\frac{1-zz}{1+zz}}$, erit

$$\sqrt{\frac{1+uu}{2}} = \frac{1}{\sqrt{(1+zz)}} \quad \text{et} \quad \sqrt{\frac{1-uu}{2}} = \frac{z}{\sqrt{(1+zz)}}.$$

..o, cum ex natura curvae lemniscatae pro puncto N coordinatae sint

$$CQ = u\sqrt{\frac{1+uu}{2}} \quad \text{et} \quad QN = u\sqrt{\frac{1-uu}{2}},$$

erit

$$CQ = \frac{u}{\sqrt{(1+zz)}} \quad \text{et} \quad QN = \frac{uz}{\sqrt{(1+zz)}} \quad \text{ideoque}$$

Quare si in A ad axem CA erigatur normalis AT, donec ductae occurrat in T, erit $AT = z = CM$.

COROLLARIUM 4

27. Ex dato ergo puncto M alterum punctum N ita capiatur tangens AT aequalis cordae CM ductaque rec puncto quaesito N secabit. Ob eandem autem rationem p producatur, donec tangenti in A occurrat in S, erit pariter

COROLLARIUM 5

28. Manifestum etiam est puncta M et N in unum posse, in quo propterea totus quadrans COA in duas pa ditur. Invenietur ergo hoc punctum O, si ponatur $u = z$,

$$z^4 + 2zz = 1 \quad \text{hincque} \quad zz + 1 = \sqrt{2};$$

prodit ergo corda $CO = \sqrt{(\sqrt{2} - 1)}$, cui simul tangens AT simul positio huius puncti O facile assignatur.

COROLLARIUM 6

29. Notato ergo hoc puncto O, quo totus quadrans C aequales CMO et ANO dividitur, erit quoque punctis M expositam definitis arc. MO = arc. ON, ita ut idem hoc arcus MN in duas partes aequales dispescat.

THEOREMA 5

30. *In curva lemniscata, cuius axis $CA = 1$ (Fig. 8), si quaecunque $CM = z$ aliaque insuper chorda applicetur*

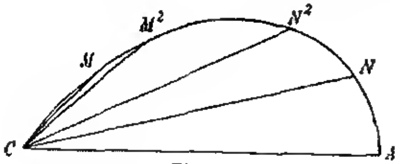


Fig. 8.

$$CM^2 = u = 2$$

erit arcus a corda ha duplo maior quam ar sublensus CM.

DEMONSTRATIO

t corda $CM = z$, erit arcus $CM = \int \frac{dz}{\sqrt{(1-z^4)}}$ similiterque ob cordam

it arcus $CM^2 = \int \frac{du}{\sqrt{(1-u^4)}}$. Quia autem est $u = \frac{2z\sqrt{(1-z^4)}}{1+z^4}$, erit

$$uu = \frac{4zz - 4z^6}{1 + 2z^4 + z^8}$$

$$\sqrt{(1-uu)} = \frac{1-2zz-z^4}{1+z^4} \quad \text{et} \quad \sqrt{(1+uu)} = \frac{1+2zz-z^4}{1+z^4},$$

$$\sqrt{(1-u^4)} = \frac{1-6z^4+z^8}{(1+z^4)^2}.$$

ifferentiando colligitur

$$du = \frac{2dz(1-z^8) - 4z^4dz(1+z^4) - 8z^4dz(1-z^4)}{(1+z^4)^2\sqrt{(1-z^4)}}$$

$$du = \frac{2dz - 12z^4dz + 2z^8dz}{(1+z^4)^2\sqrt{(1-z^4)}} = \frac{2dz(1-6z^4+z^8)}{(1+z^4)^2\sqrt{(1-z^4)}}.$$

go nanciscimur

$$\frac{du}{\sqrt{(1-u^4)}} = \frac{2dz}{\sqrt{(1-z^4)}}$$

lo arc. $CM^2 = 2$ arc. CM + Const. Cum autem posito $z = 0$ fiat ideoque ambo arcus CM et CM^2 evanescant, constans quoque ibit. Sicque sumta corda $CM^2 = u = \frac{2z\sqrt{(1-z^4)}}{1+z^4}$ erit

$$\text{arcus } CM^2 = 2 \text{ arc. } CM.$$

COROLLARIUM 1

capiatur corda $CN = \sqrt{\frac{1-zz}{1+zz}}$, erit arcus AN = arc. CM hincque CM^2 erit = 2 arc. AN. Simili modo si capiatur corda $CN^2 = \sqrt{\frac{1-uu}{1+uu}}$, N^2 = arc. CM^2 sicque etiam a vertice A erit arc. $AN^2 = 2$ arc. AN. odo obtinentur quatuor arcus inter se aequales, scilicet arc. CM, rc. AN et arc. NN^2.

COROLLARIUM 2

32. Cum autem sit

$$u = \frac{2z\sqrt{(1-z^4)}}{1+z^4}, \quad \sqrt{(1-uu)} = \frac{1-2zz-z^4}{1+z^4} \quad \text{et} \quad \sqrt{(1+uu$$

hae quatuor cordae ita habebuntur expressae, ut sit

$$CM = z, \quad CN = \sqrt{\frac{1-zz}{1+zz}}, \quad CM^2 = \frac{2z\sqrt{(1-z^4)}}{1+z^4}, \quad CN^2$$

COROLLARIUM 3

33. Conveniant ambo puncta M^2 et N^2 in curvae puncto m
quo supra vidimus esse cordam $CO = \sqrt{(\sqrt{2}-1)}$, atque ho
COA in quatuor partes aequales dispescetur in punctis A
igitur evenit, si sit $CM^2 = CN^2 = \sqrt{(\sqrt{2}-1)}$, ita ut posit
$\sqrt{(\sqrt{2}-1)} = \alpha$ habeamus

$$1 - 2zz - z^4 = \alpha + 2\alpha zz - \alpha z^4 \quad \text{seu} \quad z^4 = \frac{-2(1+\alpha}{1}$$

et

$$zz = \frac{-(1+\alpha)+\sqrt{2(1+\alpha\alpha)}}{1-\alpha} \quad \text{vel} \quad zz = \frac{-1-\sqrt{(\sqrt{2}-1}}{1-\sqrt{(\sqrt{2}}}$$

Unde colligimus

$$CM = z = \sqrt{\frac{-1-\alpha+\sqrt{2(1+\alpha\alpha)}}{1-\alpha}} \quad \text{et} \quad CN = \sqrt{\frac{-1+\alpha}{}}$$

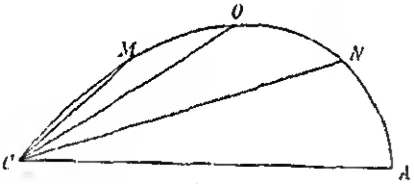
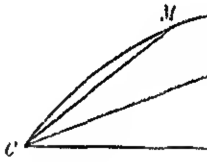


Fig. 9.



Fi

COROLLARIUM 4

34. Coalescant ambo puncta M^2 et N (Fig. 10) et punc
coibunt sicque tota curva $CMNA$ in punctis M et N t
Pro hoc ergo casu habebitur vel

$$\frac{2z\sqrt{(1-z^4)}}{1+z^4} = \sqrt{\frac{1-zz}{1+zz}} \quad \text{vel} \quad z = \frac{1-2zz-z}{1+2zz-z}$$

or dat $1 - z - 2zz - 2z^3 - z^4 + z^5 = 0$ haecque per $1 + z$ divisa $z^4 = 0$; cuius concipiantur factores

$$(1 - \mu z + zz)(1 - \nu z + zz) = 0$$

$= 2$ et $\mu\nu = -2$, unde fit $\mu - \nu = 2\sqrt{3}$ hincque

$$\mu = 1 + \sqrt{3} \quad \text{et} \quad \nu = 1 - \sqrt{3}.$$

$$z = \frac{1 + \sqrt{3} \pm \sqrt{2\sqrt{3}}}{2} = CM$$

$\frac{4\sqrt{3} \pm 2(1 + \sqrt{3})\sqrt{2\sqrt{3}}}{4}$ orietur

$$= \sqrt{\frac{1 - zz}{1 + zz}} = \sqrt{\frac{-2\sqrt{3} \mp (1 + \sqrt{3})\sqrt{2\sqrt{3}}}{4 + 2\sqrt{3} \pm (1 + \sqrt{3})\sqrt{2\sqrt{3}}}} = \sqrt{\frac{\mp\sqrt{2\sqrt{3}}}{1 + \sqrt{3}}}.$$

$$CM = \frac{1 + \sqrt{3} - \sqrt{2\sqrt{3}}}{2} \quad \text{et} \quad CN = \sqrt{\frac{\sqrt{2\sqrt{3}}}{1 + \sqrt{3}}}.$$

COROLLARIUM 5

etiam quocunque arcu CM^2 (Fig. 8, p. 94) inveniri potest eius si enim arcus illius ponatur corda $CM^2 = u$ et arcus quaesiti , erit

$$= \frac{2z\sqrt{(1 - z^4)}}{1 + z^4} \quad \text{et} \quad 1 - \frac{4zz}{uu} + 2z^4 + \frac{4z^6}{uu} + z^8 = 0,$$

concipiantur

$$(1 - \mu zz - z^4)(1 - \nu zz - z^4) = 0;$$

$\mu + \nu = \frac{4}{uu}$ et $\mu\nu = 4$; erit ergo

$$\mu - \nu = 4\sqrt{\left(\frac{1}{u^4} - 1\right)} = \frac{4}{uu}\sqrt{(1 - u^4)}$$

$$\mu = \frac{2 + 2\sqrt{(1 - u^4)}}{uu} \quad \text{et} \quad \nu = \frac{2 - 2\sqrt{(1 - u^4)}}{uu},$$

ergo

$$zz = \frac{-1 - \sqrt{(1-u^4)} + \sqrt{2(1+\sqrt{(1-}}{uu}$$

unde pro z duplex valor realis elicitur, alter

$$z = \frac{\sqrt{\left(-1-\sqrt{(1-u^4)}+\sqrt{2(1+\sqrt{(1-u^4)})}\right)}}{u} = \sqrt{(1-\sqrt{(}}$$

alter

$$z = \frac{\sqrt{\left(-1+\sqrt{(1-u^4)}+\sqrt{2(1-\sqrt{(1-u^4)})}\right)}}{u} = \sqrt{(1+\sqrt{(}}$$

COROLLARIUM 6

36. Duplex hic valor revera locum obtinet; cum
(Fig. 11) et Cm^2 duos arcus diversos CM^2 et CM^2m^2 su
z praebebit cordam arcus CM, qui est semissis arcu
ipsius z dat cordam arcus Cm, qui est semissis arcus
valor pro illo casu, posterior vero pro hoc locum h

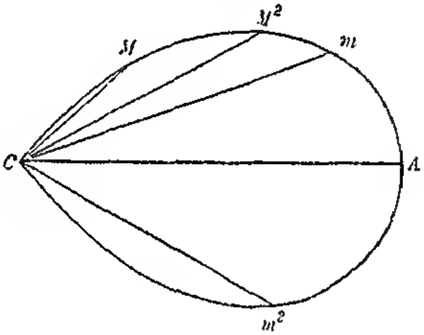
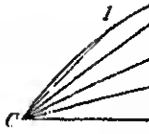


Fig. 11.



COROLLARIUM 7

37. Hoc modo etiam lemniscata CA in quinq
potest. Sit enim corda partis simplicis $C1 = z$ (Fig. 1

$$C2 = \frac{2z\sqrt{(1-z^4)}}{1+z^4} = u;$$

erit corda partis quadruplicatae

$$C4 = \frac{2u\sqrt{(1-u^4)}}{1+u^4} = \sqrt{\frac{1-zz}{1+zz}},$$

$= C1$, unde corda z definitur; qua inventa, cum sit $C2 = A3$, $= \sqrt{\frac{1-uu}{1+uu}}$.

COROLLARIUM 8

hinc posita corda cuiuspiam $= z$ reperiri possint cordae arcuum pli, octupli, sedecupli etc., manifestum est hoc modo etiam tot partes dividi posse, quarum numerus sit $2^m(1+2^n)$. In mula continentur sequentes numeri

3, 4, 5, 6, 8, 9, 10, 12, 16, 17, 18, 20, 24, 32, 33 etc.

m semper omnia divisionum puncta assignare licet.

SCHOLION

igitur sunt, quae Ill. Comes FAGNANO de curva lemniscata ob-nae ex eius inventis derivare licet. Etsi enim tantum pro-quocunque eius duplum assignare docuit, tamen hunc arcum o duplicando etiam cordae arcuum quadrupli, octupli, sede-colligentur. Namque si corda arcus simpli statuatur $= z$, u, quadrupli $= p$, octupli $= q$, sedecupli $= r$ etc., erit

$$u = \frac{2z\sqrt{(1-z^4)}}{1+z^4}$$

$$p = \frac{2u\sqrt{(1-u^4)}}{1+u^4} = \frac{4z(1+z^4)(1-6z^4+z^8)\sqrt{(1-z^4)}}{(1+z^4)^4+16z^4(1-z^4)^2}$$

$$q = \frac{2p\sqrt{(1-p^4)}}{1+p^4}$$

$$r = \frac{2q\sqrt{(1-q^4)}}{1+q^4} \quad \text{etc.}$$

arcuum multiplorum cordas ex his assignare non licet. Quem-o arcuum quorumvis multiplorum cordae exprimantur, hic in-hoc argumentum, quantum limites Analyseos id quidem per-us perficiatur. Primum quidem tentando elicui, si arcus simpli tum arcus tripli cordam fore $= \frac{z(3-6z^4-z^8)}{1+6z^4-3z^8}$; verum postea rem generaliter expediri posse intellexi.

40. *Si corda arcus simplicis CM (Fig. 13) sit $= z$*
$CM^n = u$, erit corda arcus $(n+1)$-cupli

$$CM^{n+1} = \frac{z\sqrt{\frac{1-uu}{1+uu}} + u\sqrt{\frac{1-zz}{1+zz}}}{1 - uz\sqrt{\frac{(1-uu)(1-zz)}{(1+uu)(1+zz)}}}.$$

DEMONSTRATIO

Erit ergo ipse arcus simplex

$$CM = \int \frac{dz}{\sqrt{(1-z^4)}}$$

et arcus n-cuplus

$$CM^n = \int \frac{d\cdots}{\sqrt{(1-\cdots}}$$

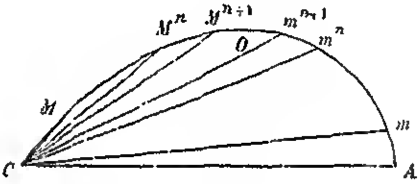


Fig. 13.

ideoque habemus
namus brevitatis

$$z\sqrt{\frac{1-uu}{1+uu}} = P$$

ut sit corda pro arcu $(n+1)$-cuplo exhibita $CM^{n+1} =$
$= s$, atque demonstrari oportet esse arcum huic corda

$$\int \frac{ds}{\sqrt{(1-s^4)}} = (n+1)\int \frac{dz}{\sqrt{(1-z^4)}} \quad \text{seu} \quad \frac{ds}{\sqrt{(1-s^4)}}$$

Cum autem sit $s = \frac{P+Q}{1-PQ}$, erit

$$ds = \frac{dP(1+QQ)+dQ(1+PP)}{(1-PQ)^2};$$

tum vero reperitur

$$1-s^4 = \frac{(1-PQ)^4-(P+Q)^4}{(1-PQ)^4} = \frac{(1+PP+QQ+PPQQ)(1-P\cdots}{(1-PQ\cdots}$$

ergo

$$\sqrt{(1-s^4)} = \frac{\sqrt{(1+PP)(1+QQ)(1-PP-QQ-4\cdots}}{(1-PQ)^2}$$

itur

$$\frac{ds}{\sqrt{(1-s^4)}}=\frac{dP\sqrt{\frac{1+QQ}{1+PP}}+dQ\sqrt{\frac{1+PP}{1+QQ}}}{\sqrt{(1-PP-QQ-4PQ+PPQQ)}},$$

essionis ergo valorem investigemus.

imo quidem est

$$PP=\frac{1+uu+zz-uuzz}{1+uu}\quad \text{et}\quad 1+QQ=\frac{1+uu+zz-uuzz}{1+zz},$$

$\frac{1+PP}{1+QQ}=\frac{1+zz}{1+uu}$ ideoque

$$\frac{ds}{\sqrt{(1-s^4)}}=\frac{dP\sqrt{\frac{1+uu}{1+zz}}+dQ\sqrt{\frac{1+zz}{1+uu}}}{\sqrt{(1-PP-QQ+PPQQ-4PQ)}}.$$

o ob

$$PP=\frac{1+uu-zz+uuzz}{1+uu}\quad \text{et}\quad 1-QQ=\frac{1+zz-uu+uuzz}{1+zz}$$

$$PP)(1-QQ)=1-P^2-Q^2+P^2Q^2=\frac{1-z^4-u^4+4uuzz+u^4z^4}{(1+zz)(1+uu)}$$

$$4PQ=\frac{4uz\sqrt{(1-z^4)(1-u^4)}}{(1+zz)(1+uu)};$$

ncluditur denominator

$$\sqrt{(1-PP-QQ+PPQQ-4PQ)}$$

$$=\frac{\sqrt{(1-z^4-u^4+4uuzz+u^4z^4-4uz\sqrt{(1-z^4)(1-u^4)})}}{\sqrt{(1+zz)(1+uu)}}=\frac{\sqrt{(1-z^4)(1-u^4)}-2uz}{\sqrt{(1+zz)(1+uu)}},$$

tinebitur

$$\frac{ds}{\sqrt{(1-s^4)}}=\frac{dP(1+uu)+dQ(1+zz)}{\sqrt{(1-z^4)(1-u^4)}-2uz}.$$

differentiando elicimus

$$dP=dz\sqrt{\frac{1-uu}{1+uu}}-\frac{2zudu}{(1+uu)\sqrt{(1-u^4)}},$$

$$dQ=du\sqrt{\frac{1-zz}{1+zz}}-\frac{2zudz}{(1+zz)\sqrt{(1-z^4)}},$$

quare ob

$$du = \frac{n\,dz\,\sqrt{(1-u^4)}}{\sqrt{(1-z^4)}}$$

erit

$$dP = dz\sqrt{\frac{1-uu}{1+uu}} - \frac{2nuzdz}{(1+uu)\sqrt{(1-}$$

$$dQ = \frac{n\,dz\,\sqrt{(1-u^4)}}{1+zz} - \frac{2uzdz}{(1+zz)\sqrt{(1}$$

unde conficitur numerator

$$dP(1+uu) + dQ(1+zz) = dz\sqrt{(1-u^4)} - \frac{2nuzdz}{\sqrt{(1-z^4)}} +$$

sive

$$dP(1+uu) + dQ(1+zz) = (n+1)dz\sqrt{(1-}$$

$$= \frac{(n+1)dz}{\sqrt{(1-z^4)}}\left(\sqrt{(1-z^4)(1-u^4)} - \right.$$

unde perspicuum est esse

$$\frac{ds}{\sqrt{(1-s^4)}} = \frac{(n+1)dz}{\sqrt{(1-z^4)}}$$

et

$$\text{arc. } CM^{n+1} = (n+1) \text{ arc. } C$$

Q. E. D.

COROLLARIUM 1

41. Si a vertice A abscindantur arcus Am, CM^n, CM^{n+1} respective aequales, erit Cm corda co corda complementi arcus CM^n, Cm^{n+1} corda comple autem ob cordas $CM = z$, $CM^n = u$, $CM^{n+1} = s$ c

$$Cm = \sqrt{\frac{1-zz}{1+zz}}, \quad Cm^n = \sqrt{\frac{1-uu}{1+uu}}, \quad Cm$$

Cum autem sit

$$s = \frac{z\sqrt{\frac{1-uu}{1+uu}} + u\sqrt{\frac{1-zz}{1+zz}}}{1 - zu\sqrt{\frac{(1-uu)(1-zz)}{(1+uu)(1+zz)}}} = \frac{P}{1-}$$

erit

$$\sqrt{\frac{1-ss}{1+ss}} = \sqrt{\frac{1-PP-QQ-4PQ+PPQQ}{(1+PP)(1+QQ)}} = \sqrt{}$$

ad hanc formam reducitur

$$\sqrt{\frac{1-ss}{1+ss}} = \frac{\sqrt{\frac{(1-zz)(1-uu)}{(1+zz)(1+uu)}}-uz}{1+uz\sqrt{\frac{(1-zz)(1-uu)}{(1+zz)(1+uu)}}}.$$

COROLLARIUM 2

. Si igitur ponatur

corda arcus simplicis $= z$, corda complementi $= Z$,
corda arcus n-cupli $= u$, corda complementi $= U$,

$$Z = \sqrt{\frac{1-zz}{1+zz}} \quad \text{et} \quad U = \sqrt{\frac{1-uu}{1+uu}},$$

$$\text{corda arcus } (n+1)\text{-cupli} = \frac{zU+uZ}{1-zuZU},$$

$$\text{corda complementi} = \frac{ZU-zu}{1+zuZU}.$$

COROLLARIUM 3

Inventio ergo cordarum arcuum quorumvis multiplorum una cum complementi ita se habebit:

Corda arcus	Corda complementi
simpli $= a$	simpli $= A$
dupli $= b = \frac{2aA}{1-aaAA}$	dupli $= \frac{AA-aa}{1+aaAA} = B$
tripli $= c = \frac{aB+bA}{1-abAB}$	tripli $= \frac{AB-ab}{1+abAB} = C$
quadrupli $= d = \frac{aC+cA}{1-acAC}$	quadrupli $= \frac{AC-ac}{1+acAC} = D$
quintupli $= e = \frac{aD+dA}{1-adAD}$	quintupli $= \frac{AD-ad}{1+adAD} = E$
etc.	etc.

COROLLARIUM 4

Simili modo si corda arcus m-cupli sit $= r$, corda complementi $= R$ a arcus n-cupli $= s$ eiusque corda complementi $= S$, ut sit

$$R = \sqrt{\frac{1-rr}{1+rr}} \quad \text{et} \quad S = \sqrt{\frac{1-ss}{1+ss}},$$

erit corda arcus $(m+n)$-cupli $= \frac{rS+sR}{1-rsRS}$ et corda
Quin etiam sumendo pro n numerum negativum, q
sui negativum, corda differentiae illorum arcuum exh
corda arcus $(m-n)$-cupli $= \frac{rS-sR}{1+rsRS}$ et corda comp

COROLLARIUM 5

45. Sumtis ergo denominationibus, quae in cor
erit quoque

$$d = \frac{2bB}{1-bbBB} \quad \text{et} \quad D = \frac{BB-}{1+bb}$$
$$c = \frac{bC+cB}{1-bcBC} \quad \text{et} \quad E = \frac{BC-}{1+bc}$$

COROLLARIUM 6

46. Ex his colligitur, si corda arcus simplicis st
darum in corollario 3 adhibitarum fore

$$a = z \qquad A = \sqrt{\frac{1-zz}{1+zz}}$$
$$b = \frac{2z\sqrt{(1-z^4)}}{1+z^4} \qquad B = \frac{1-2zz-}{1+2zz-}$$
$$c = \frac{z(3-6z^4-z^8)}{1+6z^4-3z^8} \qquad C = \frac{(1+z^4)^2-}{(1+z^4)^2+}$$
$$d = \frac{4z(1+z^4)(1-6z^4+z^8)\sqrt{(1-z^4)}}{(1+z^4)^4+16z^4(1-z^4)^2} \qquad D = \frac{(1-6z^4+}{(1-6z^4+}$$

SCHOLION 1

47. Ratio compositionis formularum $\frac{rS+sR}{1-rsRS}$ e
notari meretur, quod similis est regulae, qua tangen
duorum angulorum definiri solet. Si enim sit $rS =$
erit $\frac{rS+sR}{1-rsRS} = \text{tang.}(\alpha+\beta)$ et pro differentia i
$\frac{rS-sR}{1+rsRS} = \text{tang.}(\alpha-\beta)$. Similique modo si po
$rs = \text{tang.}\,\delta$, erit

$$\frac{RS-rs}{1+rsRS} = \text{tang.}(\gamma-\delta) \quad \text{et} \quad \frac{RS+rs}{1-rsRS} =$$

modus autem ista compositionis ratio repraesentabitur, si ponatur m-cupli $r = M\sin.\mu$, corda complementi $R = M\cos.\mu$, corda pli $s = N\sin.\nu$, corda complementi $S = N\cos.\nu$; tum enim erit

$$\text{corda arcus } (m+n)\text{-cupli} = \frac{MN\sin.(\mu+\nu)}{1 - M^2N^2\sin.\mu\sin.\nu\cos.\mu\cos.\nu}$$

$$\text{corda eius complementi} = \frac{MN\cos.(\mu+\nu)}{1 + M^2N^2\sin.\mu\sin.\nu\cos.\mu\cos.\nu}$$

$$\text{corda arcus } (m-n)\text{-cupli} = \frac{MN\sin.(\mu-\nu)}{1 + M^2N^2\sin.\mu\sin.\nu\cos.\mu\cos.\nu}$$

$$\text{corda eius complementi} = \frac{MN\cos.(\mu-\nu)}{1 - M^2N^2\sin.\mu\sin.\nu\cos.\mu\cos.\nu}.$$

autem sit $1 - rr - RR = rrRR$, erit $1 - MM = M^4\sin.\mu^2\cos.\mu^2$ i

$$M^2\sin.\mu\cos.\mu = \sqrt{(1 - MM)} \quad \text{et} \quad N^2\sin.\nu\cos.\nu = \sqrt{(1 - NN)},$$

istarum formularum denominatores abibunt in

$$1 - \sqrt{(1 - MM)(1 - NN)} \quad \text{et} \quad 1 + \sqrt{(1 - MM)(1 - NN)}.$$

terea vero ex illa aequatione $1 - MM = M^4\sin.\mu^2\cos.\mu^2$ fit

$$\frac{1}{MM} = \frac{1}{2} + \frac{1}{2}\sqrt{(1 + \sin.2\mu\sin.2\mu)}$$

$\sin.2\mu = 2\sin.\mu\cos.\mu$. Verum hinc illae formulae non concin unt.

SCHOLION 2

48. Ex his observationibus calculus integralis non contemnenda aug equitur, siquidem hinc plurimarum aequationum differentialium int culares exhibere valemus, quarum integratio in genere vix sperari proposita aequatione differentiali

$$\frac{du}{\sqrt{(1-u^4)}} = \frac{dz}{\sqrt{(1-z^4)}},$$

terquam quod casus integralis $u = z$ per se est obvius, novimus ei facere $u = -\sqrt{\frac{1-zz}{1+zz}}$. In genere igitur cum integratio constantem am, puta C, involvat, erit u aequalis functioni cuipiam quantitatum tamen nihilominus ita erit comparata, ut pro certo quodam ip ro fiat $u = z$ itemque pro alio quodam ipsius C valore $u = -\sqrt{}$

expressionem algebraicam adeo simplicem convertunt.

Simili modo proposita hac aequatione

$$\frac{du}{\sqrt{(1-u^4)}} = \frac{2dz}{\sqrt{(1-z^4)}}$$

duos habemus valores, quos ei satisfacere novimus,

$$u = \frac{2z\sqrt{(1-z^4)}}{1+z^4} \quad \text{et} \quad u = \frac{-1+2zz+z^4}{1+2zz-z^4}$$

pariterque geminos valores exhibere docuimus, qui in gener satisfaciant

$$\frac{mdu}{\sqrt{(1-u^4)}} = \frac{ndz}{\sqrt{(1-z^4)}},$$

unde via ad harum formularum integralia generalia inven praeparata videtur.

Deinde quae supra de ellipsi et hyperbola sunt allata, tionum differentialium integrationes speciales suppeditant.

Proposita enim ex § 3 hac aequatione

$$dx\sqrt{\frac{1-nxx}{1-xx}} + du\sqrt{\frac{1-nuu}{1-uu}} = (xdu+udx)\sqrt{}$$

novimus ei satisfacere hanc aequationem integralem

$$1 - nxx - nuu + nuuxx = 0.$$

Isti autem aequationi ex § 5 petitae

$$dx\sqrt{\frac{1-nxx}{1-xx}} + du\sqrt{\frac{1-nuu}{1-uu}} = n(xdu+udx)$$

satisfacere inventa est haec aequatio

$$1 - xx - uu + nuuxx = 0.$$

Deinde sequenti aequationi ex hyperbola § 14 petitae

$$dx\sqrt{\frac{nxx-1}{xx-1}} + du\sqrt{\frac{nuu-1}{uu-1}} = (xdu+udx)\sqrt{}$$

satisfacit quoque

$$1 - nxx - nuu + nuuxx = 0,$$

em cum priore ex ellipsi petita congruit, cum sit

$$\sqrt{\frac{nxx-1}{xx-1}} = \sqrt{\frac{1-nxx}{1-xx}}.$$

n facile concludere licet, huic aequationi

$$dx\sqrt{\frac{f-gxx}{h-kxx}} + du\sqrt{\frac{f-guu}{h-kuu}} = (xdu+udx)\sqrt{\frac{g}{h}}$$

hanc integralem specialem

$$fh - gh(xx+uu) + gkxxuu = 0,$$

aequationi alteri

$$dx\sqrt{\frac{f-gxx}{h-kxx}} + du\sqrt{\frac{f-guu}{h-kuu}} = (xdu+udx)\frac{g}{\sqrt{fk}}$$

hanc integralem specialem

$$fh - fk(xx+uu) + gkxxuu = 0.$$

r ideo proponenda censui, quod ansam mihi praebere videntur sul
yseos ulterius excolendi.

SPECIMEN NOVAE METHODI C
QUADRATURAS ET RECTIFIC
ALIASQUE QUANTITATES TRAN
INTER SE COMPARAN

Commentatio 263 indicis ENESTROEMIAN
Novi Commentarii academiae scientiarum Petropolitanae 7 (17
Summarium (Commentationum 263 et 261) ibide

SUMMARIUM

Principio monendus est lector rogandaque errori typoth
posterior ordine dissertatio[1]) priori est anteposita. Culpam hanc
utramque dissertationem simul considerabimus et consueta nobi
stitum sit, dicemus. Versatur methodus a Cel. Auctore propo
circa quantitates transcendentes seu eiusmodi quantitates in lin
nullo modo algebraice exprimi possunt. Semper consideratio li
se videatur, tam Geometriam quam Analysin pulcerrimis inventi
enim Geometrae lineas curvas contemplari coeperunt, statim om
ut tam spatia ab iis inclusa quam ipsam earum longitudinem
gationum prior circa curvarum quadraturas, altera circa earur
batur. Quoniam vero neutrum in circulo praestari poterat, etsi
est simplicissima, eo maiori studio in eiusmodi lineas curvas in
turam, hoc est spatii iis inclusi dimensionem, vel rectificatio
aequalis assignari debet, admitterent. Interim tamen etiam in
quadratura circuli investiganda frustra desudarunt, praeter
inventa sunt consecuti, quibus idem usu venit, quod Alchimis

---

1) L. EULERI Commentatio 261 (indicis ENESTROEMIANI); vi

um praeparatione occupati, etsi voto suo exciderunt, plurima saluberrima rem medicinae contulerunt. Post inventam autem Analysin infinitorum summum s praecipue in quadrandis et rectificandis lineis curvis est consumtum, uberrimos it, quibus plures methodos satis sublimes, quarum usus per universam M simus existit, acceptas referre debemus. Quare haud minores fructus ab eorum are licet, qui in comparatione linearum curvarum, quae per se vel quadratur cationem respuunt, exquirenda laborant, in quo negotio certe profundissima An sunt adeunda, ita ut, qui hic quicquam praestiterit, is plurimum in hac isse sit censendus.

Huc sine dubio referenda est nova methodus a Cel. Auctore excogitata, cu erabilium curvarum, quarum rectificatio omnes vires Analyseos transcendit, arc nparare docet. *Pro iis quidem curvis, quarum rectificatio* ope circuli vel l n expediri potest, hoc cognitis methodis praestari potest, sed totum negotium s beneficio huius methodi conficitur, quemadmodum ex specimine posteriore lu t, ubi comparationem arcuum circularium, aliunde quidem satis cognitum, et licorum mira simplicitate exequitur, ut iam hinc summa utilitas huius a eluceat.

n altero autem specimine, quod hic primo loco extat, hanc methodum potissi in accommodatam conspicimus, cuius lineae rectificationem neque ad arcus ci logarithmos revocari posse inter Geometras satis superque constat. Neque e *rva binos arcus dissimiles, qui inter se sint aequales, abscindere licet*, ex qu mirum videbitur data huius curvae arcu quocunque semper alium arcum et n puncto terminatum exhiberi posse, qui ab illo differat quantitate geometrice um hoc ne in circulo quidem praestari queat. Si enim differentia inter duc res geometrice assignari posset, eo ipso rectificatio circuli absoluta haberet autem haec ratio longe aliter est comparata, cum innumerabilibus modis di os arcus ellipticos definiri possit. Simili modo, proposito arcu ellipseos quo *o quovis puncto arcum abscindere licet, qui ab illius duplo vel triplo vel ali* lo atque etiam submultiplo quantitate geometrice assignabili differat. Im otest, ut haec differentia prorsus evanescat sicque bini arcus elliptici datam em tenentes exhiberi queant, dummodo ratio illa non sit aequalitatis, quippe q rcus prodeunt inter se similes, in quo nihil singulare habetur. Cuncta aut mata, quae Cel. Auctor hic pro Ellipsi expedivit, simili plane modo etiam pro atque infinitis aliis lineis curvis multo magis complicatis resolvi posse ma x quo haec methodus omni Geometrarum attentione et uberiori evolutione di r.

sum de comparatione arcuum ellipsis, hyperbolae et curv
latius mihi quidem patere statim sunt visa. Cum e
consuetis eiusmodi tantum curvarum arcus inter se comp
rectificatio vel a quadratura circuli vel a logarithmis p
quantitates, etsi sunt transcendentes, tamen ita iam in
ius quoddam civitatis sunt adeptae, ut perinde atque
queant, maxima certe attentione erat dignum, quod a F
et ellipsi arcus sint assignati, quorum differentia sit alg
autem eiusmodi arcus, qui adeo inter se sint aequale
rationem, propterea quod harum curvarum rectificatio n
circuli neque ad logarithmos reduci queat. Hinc certe
transcendentium insigne lumen accenderetur, si modo v
usus, certam methodum suppeditaret in huiusmodi inv
progrediendi; sed quia tota substitutionibus precario
fortuito adhibitis nititur, parum inde utilitatis in Analy
iam notavi integrationes, quas operatio FAGNANIANA con
particulares neque idcirco methodum certam, a qua
suppeditare. Interim tamen ea amplissimum campum
quo ulterius excolendo Geometrae vires suas summo
ad insigne Analyseos incrementum.

Res autem huc redit, ut propositis duabus formu
et $\int Y dy$ non integrabilibus, ubi X sit functio quaepiam
eiusmodi relatio inter variabiles x et y definiatur, ut i
se fiant aequales vel datam rationem teneant, vel ut
assignabilem obtineant. Quae investigatio cum latissin
insignes in se continet casus iam pridem non sine ma
mento evolutos; huc enim referenda sunt, quae de
circularium, de lunulis quadrabilibus, de zonis cyclo
tum vero de arcubus parabolicis, qui vel datam inter se
differentiam algebraicam habeant, a geometris sunt tra
investigatio a Cel. Ioh. BERNOULLI[2]) ad parabolas cubic

1) L. EULERI Commentatio 252 (indicis ENESTROEMIANI); vide
2) IOH. BERNOULLI, *Investigatio algebraica arcuum parabolicorum a*
bentium. Demonstratio isochronismi descensuum in cycloide etc., Acta er
T. 1, p. 242; *Theorema universale rectificationi linearum curvarum inservie*
tas. Cubicalis primariae arcuum mensura etc., Acta erud. 1698, p. 4

..., sed quia ratio, qua usus est, nulla certa methodo nitebatur, ...u fere penitus caruit. Hoc quoque pertinet, quod multo ante iam ... HUGENIUS[1]) in *Horologio oscillatorio* exposuerat, ubi proposito ... elliptico compresso seu revolutione circa axem minorem genito ...ecuit conoides hyperbolicum, ita ut summa utriusque superficiei ...iberi possct, cum tamen neutra superficies seorsim cum circulo ...queat. Quae inventio iam tum summis Geometris maxime memo... est; atque BERNOULLIUS in litteris ad LEIBNIZIUM[2]) datis dolet hanc ... nulla certa methodo inniti, ex qua plura huius generis inventa ...ceant; interim quia superficies tam illius sphaeroidis elliptici quam ...perbolici a logarithmis pendet, reductio utriusque iunctim sumtae ... simili modo perfici potest, quo in parabola arcus algebraicam ...ifferentiam assignari solent. Inprimis autem hoc loco non est ...um TSCHIRNHAUSIUM[3]) quondam methodum a se inventam iactasse, ...icio curvarum quarumcunque non rectificabilium arcus ita inter se ... possent, ut differentia fiat algebraica; sed praeterquam, quod ...suam nunquam aperuerit, manifestum est eum paralogismo quodam ...btum ut saepius alias, cum certum sit rem ita generaliter omnino ...u posse; neque ergo TSCHIRNHAUSIUS putandus est quicquam eorum ...mo vel tum circa comparationem curvarum sunt inventa vel adhuc ...ntur.

...en igitur quoddam methodi huiusmodi quaestiones solvendi hic ...nstitui, quod non obscure maiores progressus in hac re pro...otur; atque cum non solum difficillimum sit propositis in genere ...ormulis integralibus quaesitam inter variabiles relationem eruere, ...hoc saepissime omnino ne fieri quidem possit, ordine inverso rem ..., ut assumta binarum variabilium relatione inde ipsas formulas ...nvestigarem, quae per hanc relationem inter se comparari possent. ...odus cum facillime procedat, ad multo subliminora perducere posse ...ae aliis methodis plane sint impervia; hac enim methodo non

1) ... HUYGENS (1629—1695), *Horologium oscillatorium sive de motu pendulorum ad horo... ...monstrationes geometricae*, Parisiis 1673; *Opera varia* Vol. 1, 1724, p. 15, imprimis ... A. K.

2) ... EULERUS errasse videtur; cf. IOH. BERNOULLI, *Meditatio de dimensione linearum cur... ...ulares*, Acta erud. 1695, p. 374; *Opera omnia* T. 1, p. 142. A. K.

3) ... TSCHIRNHAUS (1651—1708), *Nova et singularis geometriae promotio circa dimensio... ...um curvarum*, Acta erud. 1695, p. 489. A. K.

solum ea, quae habet FAGNANUS, facili negotio ac si
assecutus, sed etiam multo ampliora atque illustriora
nimis particulariter definiverat, ego satis universalit
calculus, quo sum usus, ita comparatus est, ut, quoni
singulares complectitur, viam ad multo subliminiora sto

Tum vero quanquam variabilium mutua relatio
definiri potest, quoties integratio utriusque formulae
quadratura circuli vel a logarithmis pendet, tamen
sine molesto calculo perficitur, dum partes vel arcus
mos continentes se mutuo destruere debent, quemadm
tione arcuum parabolicorum abunde perspicitur. Per
hae difficultates cunctae penitus evanescunt ac fere
comparationes tam in circulo quam in parabola ab
dubio non exigua vis huius methodi sita esse censend
multo facilius ea, quae aliis methodis iam sunt erut
ad eiusmodi investigationes manuducat, in quibus ali
praestiturae. Quam ob rem hoc quidem loco istam
eos casus applicabo, qui etiam aliis methodis, sed m
solent, quo, cum principia, quibus innititur, hac occ
ceps facilius eius applicationem ad quaestiones subli
Quoniam igitur mihi a relatione inter binas variabiles
stituo, ordiendum est, a simplicioribus incipiam ac
modi, quae ad similes formulas integrales perducant,
similes sint prodituræ functiones ipsarum x et y. F
hinc natae ob similitudinem quantitates transcendente
lineam curvam pertinentes, deinceps autem ad form
quae ad diversas curvas pertineant, sum progressurus

RELATIO PRIMA INTER BINAS VARIAB

$$0 = \alpha + \gamma(xx + yy) + 2\delta xy$$

1. Si hinc seorsim valores x et y extrahantur,

$$y = \frac{-\delta x \pm \sqrt{((\delta\delta - \gamma\gamma)xx - \alpha\gamma)}}{\gamma},$$

$$x = \frac{-\delta y \pm \sqrt{((\delta\delta - \gamma\gamma)yy - \alpha\gamma)}}{\gamma},$$

ubi quovis casu dispiciendum est, utrum signum qu

eri enim potest, ut in utraque formula vel signa paria vel
n habeant, dum alterutrum arbitrio nostro plane relinquitur;
o inprimis natura variabilium x et y, utrum affirmative an
iantur, spectari debet.

ntur brevitatis gratia membra irrationalia

$$\delta - \gamma\gamma)xx - \alpha\gamma) = P \quad \text{et} \quad \pm \sqrt{((\delta\delta - \gamma\gamma)yy - \alpha\gamma)} = Q,$$

$$y = \frac{-\delta x + P}{\gamma} \quad \text{et} \quad x = \frac{-\delta y + Q}{\gamma},$$

$$P = \gamma y + \delta x \quad \text{et} \quad Q = \gamma x + \delta y,$$

casu facile colligere licet, utrum quantitates P et Q habiturae
ffirmativos an negativos.

entiebur iam aequatio assumta eritque

$$dx(\gamma x + \delta y) + dy(\gamma y + \delta x) = 0$$

$- \delta y = Q$ et $\gamma y + \delta x = P$ habebitur haec aequatio

$$Qdx + Pdy = 0 \quad \text{sive} \quad \frac{dx}{P} + \frac{dy}{Q} = 0.$$

o pro P et Q valoribus huic aequationi integrali

$$\int \frac{dx}{\sqrt{((\delta\delta - \gamma\gamma)xx - \alpha\gamma)}} + \int \frac{dy}{\sqrt{((\delta\delta - \gamma\gamma)yy - \alpha\gamma)}} = \text{Const.}$$

io inter variabiles x et y assumta.

amus haec accuratius, et quo facilius applicatio fieri queat,

$$-\alpha\gamma = Ap \quad \text{et} \quad \delta\delta - \gamma\gamma = Cp,$$

$$\int \frac{dx}{\sqrt{(A + Cxx)}} + \int \frac{dy}{\sqrt{(A + Cyy)}} = \text{Const.},$$

eritque

$$\alpha = -\frac{Ap}{\gamma} \quad \text{et} \quad \delta = \sqrt{(Cp + \gamma\gamma)}$$

sicque quantitates p et γ arbitrio nostro relinquuntur.

5. Statuatur ergo $\gamma = A$ et $p = Akk$, ita ut k sit nov stans a nostro arbitrio pendens, eritque

$$\alpha = -Akk, \quad \gamma = A \quad \text{et} \quad \delta = \sqrt{A(A + Ckk)}$$

et aequatio canonica nostrae aequationi integrali satisfaciens

$$0 = -Akk + A(xx + yy) + 2xy\sqrt{A(A + Ckk)}$$

seu

$$y = \frac{-x\sqrt{(A + Ckk)} + k\sqrt{(A + Cxx)}}{\sqrt{A}}$$

et

$$x = \frac{-y\sqrt{(A + Ckk)} + k\sqrt{(A + Cyy)}}{\sqrt{A}}.$$

6. Si $\sqrt{(A + Cyy)}$ negative capiatur itemque $\sqrt{A}$, tum differentialis

$$\frac{dx}{\sqrt{(A + Cxx)}} = \frac{dy}{\sqrt{(A + Cyy)}}$$

integralis erit

$$0 = -Akk + A(xx + yy) - 2xy\sqrt{A(A + Ckk)}$$

ideoque vel

$$y = \frac{x\sqrt{(A + Ckk)} - k\sqrt{(A + Cxx)}}{\sqrt{A}}$$

vel

$$x = \frac{y\sqrt{(A + Ckk)} + k\sqrt{(A + Cyy)}}{\sqrt{A}}.$$

7. Quia ergo aequatio integralis constantem in se com in differentiali non inest, indicio hoc est integralem esse co differentiali nulla alia satisfacit integralis, nisi quae in form hendatur. Atque haec est integratio principalis, ad quam y assumta perducit.

utem derivari possunt innumerabiles aliae integrationes. Si enim eiusmodi functiones ipsarum x et y, ut vi relationis assumtae dem relatio satisfaciet quoque huic aequationi differentiali

$$\frac{X\,dx}{\sqrt{(A + Cxx)}} = \frac{Y\,dy}{\sqrt{(A + Cyy)}}.$$

n modis huiusmodi functiones aequales exhiberi possunt ex x et y inventis.

autem haec investigatio latius pateat et X et Y sint functiones non assumo inter se aequales, eiusmodi autem pro iis valores

$$\frac{X\,dx}{\sqrt{(A + Cxx)}} - \frac{Y\,dy}{\sqrt{(A + Cyy)}} = dV$$

ns V prodeat algebraica, si scilicet relatio § 6 tradita locum

igitur sit $\frac{dy}{\sqrt{(A + Cyy)}} = \frac{dx}{\sqrt{(A + Cxx)}}$, erit

$$\frac{(X - Y)dx}{\sqrt{(A + Cxx)}} = dV$$

$$= k\sqrt{A}(A + Cxx) = \gamma y + \delta x = Ay + x\sqrt{A}(A + Ckk)$$

$\sqrt{A}$ negativo erit

$$\sqrt{(A + Cxx)} = \frac{x}{k}\sqrt{(A + Ckk)} - \frac{y}{k}\sqrt{A},$$

$$\frac{(X - Y)k\,dx}{x\sqrt{(A + Ckk)} - y\sqrt{A}} = dV.$$

sit porro ex aequatione differentiata

$$x(Ax - y\sqrt{A}(A + Ckk)) = dy(x\sqrt{A}(A + Ckk) - Ay),$$

ponatur $xy = u$; erit $dy = \frac{du}{x} - \frac{ydx}{x}$, quo valore substi

$$dx\left(Ax - \frac{Ayy}{x}\right) = \frac{du}{x}(x\sqrt{A(A + Ckk)} -$$

seu

$$\frac{dx}{x\sqrt{(A + Ckk)} - y\sqrt{A}} = \frac{du}{(xx - yy)\sqrt{A}}$$

sicque erit

$$dV = \frac{kdu}{\sqrt{A}} \cdot \frac{X - Y}{xx - yy}.$$

12. Quoties ergo $\frac{X - Y}{xx - yy}$ eiusmodi functio ipsius u, integrabilis, toties valor quantitatis V algebraice exhibe evenit, quoties X et Y fuerint potestates parium expone propterea cum sit ex aequatione assumta

$$xx + yy = kk + \frac{2u}{A}\sqrt{A(A + Ckk)}.$$

13. Ponatur ergo $X = x^n$ et $Y = y^n$; erit posito n

$$\frac{X - Y}{xx - yy} = 1 \quad \text{et} \quad dV = \frac{kdu}{\sqrt{A}}$$

ideoque

$$V = \frac{ku}{\sqrt{A}} + \text{Const.} = \frac{kxy}{\sqrt{A}} + \text{Const.}$$

Quam ob rem habebitur

$$\int \frac{xxdx}{\sqrt{(A + Cxx)}} - \int \frac{yydy}{\sqrt{(A + Cyy)}} = \text{Const.} +$$

14. Sit iam $n = 4$ eritque

$$\frac{X - Y}{xx - yy} = xx + yy = kk + \frac{2u}{A}\sqrt{A(A +}$$

unde

$$dV = \frac{kdu}{A}(kk\sqrt{A} + 2u\sqrt{(A + Ckk)})$$

ergo

$$V = \frac{ku}{A}(kk\sqrt{A} + u\sqrt{(A + Ckk)}).$$

u erit

$$\ldots x x) - \int \frac{y^4 dy}{\sqrt{(A + Cyy)}} = \text{Const.} + \frac{kxy}{A}(kk\sqrt{A} + xy\sqrt{(A + Ckk)})$$

o ulterius progredi licet.

gitur coniungendis si fuerit

$$xx + yy = kk + 2xy\sqrt{\left(1 + \frac{C}{A}kk\right)}$$

$$y = \frac{x\sqrt{(A + Ckk)} - k\sqrt{(A + Cxx)}}{\sqrt{A}},$$

$$x = \frac{y\sqrt{(A + Ckk)} + k\sqrt{(A + Cyy)}}{\sqrt{A}},$$

nter x et y satisfaciet huic aequationi integrali

$$\int \frac{dx(\mathfrak{A} + \mathfrak{B}xx + \mathfrak{C}x^4)}{\sqrt{(A + Cxx)}} - \int \frac{dy(\mathfrak{A} + \mathfrak{B}yy + \mathfrak{C}y^4)}{\sqrt{(A + Cyy)}}$$

$$= \text{Const.} + \frac{\mathfrak{B}kxy}{\sqrt{A}} + \frac{\mathfrak{C}kxy}{\sqrt{A}}\left(kk + xy\sqrt{\left(1 + \frac{C}{A}kk\right)}\right)$$

istarum formularum integralium algebraice assignari potest.

ATIO SECUNDA INTER BINAS VARIABILES x ET y

$$0 = \alpha + 2\beta(x + y) + \gamma(xx + yy) + 2\delta xy$$

iam, uti in praecedentibus deprehendimus, ambiguitas signorum arbitrio nostro pendet, dummodo eius ratio in conclusionibus e habeatur, si ad differentiam binarum formularum integralium mus, extrahendo radices habebimus

$$y = \frac{-\beta - \delta x - \sqrt{(\beta\beta - \alpha\gamma + 2\beta(\delta - \gamma)x + (\delta\delta - \gamma\gamma)xx)}}{\gamma},$$

$$x = \frac{-\beta - \delta y + \sqrt{(\beta\beta - \alpha\gamma + 2\beta(\delta - \gamma)y + (\delta\delta - \gamma\gamma)yy)}}{\gamma}.$$

17. Statuamus brevitatis gratia has formulas irrationales

$$\sqrt{(\beta\beta - \alpha\gamma + 2\beta(\delta - \gamma)x + (\delta\delta - \gamma\gamma)xx)} = P,$$
$$\sqrt{(\beta\beta - \alpha\gamma + 2\beta(\delta - \gamma)y + (\delta\delta - \gamma\gamma)yy)} = Q$$

ue

$$-P = \beta + \gamma y + \delta x \quad \text{et} \quad Q = \beta + \gamma x + \delta y,$$

oliciuntur istae relationes

$$P + Q = (\gamma - \delta)(x - y),$$
$$\gamma P + \delta Q = \beta(\delta - \gamma) + (\delta\delta - \gamma\gamma)y,$$
$$\delta P + \gamma Q = \beta(\gamma - \delta) - (\delta\delta - \gamma\gamma)x.$$

18. Aequatio autem proposita differentiata dat

$$dx(\beta + \gamma x + \delta y) + dy(\beta + \gamma y + \delta x) = 0$$

$$Qdx - Pdy = 0,$$

e oritur

$$\frac{dx}{P} = \frac{dy}{Q} \quad \text{seu} \quad \int\frac{dx}{P} - \int\frac{dy}{Q} = \text{Const.},$$

ergo aequationi integrali satisfacit relatio proposita indeque valores
y extracti.

19. Ut hinc simili modo alias integrationes obtineamus, sint iterum
Y functiones similes ipsarum x et y ac posito

$$\frac{Xdx}{P} - \frac{Ydy}{Q} = dV$$

niantur hae functiones ita, ut V prodeat quantitas algebraica si
eatur

$$\int\frac{Xdx}{P} - \int\frac{Ydy}{Q} = V + \text{Const.}$$

20. Cum igitur sit $\frac{dy}{Q} = \frac{dx}{P}$, erit

$$dV = \frac{(X - Y)dx}{P} \quad \text{seu} \quad dV = \frac{-dx(X - Y)}{\beta + \gamma y + \delta x}.$$

$= u$ ideoque $dy = \frac{du}{x} - \frac{y\,dx}{x}$; erit pro aequatione differentiali

$$\ldots \gamma x + \delta y) + \frac{du}{x}(\beta + \gamma y + \delta x) - \frac{y\,dx}{x}(\beta + \gamma y + \delta x) = 0$$

$$\ldots x(\beta x - \beta y + \gamma xx - \gamma yy) + du(\beta + \gamma y + \delta x) = 0.$$

... e hinc pro dx substituto habebitur

$$dV = \frac{du(X - Y)}{(x-y)(\beta + \gamma(x+y))}.$$

... ulterius $x + y = t$; erit $xx + yy = tt - 2u$ et aequatio assumta ... n abibit

$$0 = \alpha + 2\beta t + \gamma tt + 2(\delta - \gamma)u,$$

... tiando fit

$$dt(\beta + \gamma t) = (\gamma - \delta)du$$

$$\frac{du}{\beta + \gamma t} = \frac{dt}{\gamma - \delta}.$$

... igitur simpliciori modo obtinetur

$$dV = \frac{dt(X - Y)}{(\gamma - \delta)(x - y)},$$

... X et Y fuerint potestates ipsarum x et y, tum fractionem ... u ideoque et per solum t ob

$$u = \frac{\alpha + 2\beta t + \gamma tt}{2(\gamma - \delta)}$$

... mi posse.

... go $X = x^n$ et $Y = y^n$ ac ponatur primo $n = 1$; erit $\frac{X - Y}{x - y} = 1$ et ... do fit $V = \frac{t}{\gamma - \delta}$. Quocirca pro hoc casu erit

$$\int \frac{x\,dx}{P} - \int \frac{y\,dy}{Q} = \text{Const.} + \frac{x + y}{\gamma - \delta},$$

... ationi integrali satisfit per relationem inter x et y assumtam.

24. Sit $n = 2$ eritque $\frac{x^2 - y^2}{x - y} = x + y = t$,

$$dV = \frac{tdt}{\gamma - \delta} \quad \text{et} \quad V = \frac{tt}{2(\gamma - \delta)} = \frac{(x + y)^2}{2(\gamma - \delta)}.$$

Hoc ergo casu habebitur

$$\int \frac{xxdx}{P} - \int \frac{yydy}{Q} = \text{Const.} + \frac{(x + y)^2}{2(\gamma - \delta)}.$$

25. Si ulterius progredi lubeat, ponatur $n = 3$ eritque

$$\frac{x^3 - y^3}{x - y} = xx + xy + yy = tt - u = \frac{(\gamma - 2\delta)tt - 2\beta t - \alpha}{2(\gamma - \delta)}$$

et

$$V = \frac{\frac{1}{3}(\gamma - 2\delta)t^3 - \beta tt - \alpha t}{2(\gamma - \delta)^2}$$

sicque erit

$$\int \frac{x^3dx}{P} - \int \frac{y^3dy}{Q} = \text{Const.} + \frac{(\gamma - 2\delta)(x + y)^3 - 3\beta(x + y)^2 - 3\alpha(

$$p = \frac{(AC - BB)kk}{C} \quad \text{et} \quad \beta = \frac{B}{C}(\delta + \gamma)$$

$$\alpha = \frac{2BB}{CC}(\gamma + \delta) - \frac{(AC - BB)^2 kk}{CC(A + Bk)}$$

TIO TERTIA INTER BINAS VARIABILES x ET y

$$0 = \alpha + mxx + nyy + 2\delta xy$$

endo utramque radicem habebitur

$$y = \frac{-\delta x + \sqrt{((\delta\delta - mn)xx - \alpha n)}}{n},$$

$$x = \frac{-\delta y - \sqrt{((\delta\delta - mn)yy - \alpha m)}}{m};$$

$((\delta\delta - mn)xx - \alpha n) \quad \text{et} \quad Q = \sqrt{((\delta\delta - mn)yy - \alpha m)}$

$$P = \delta x + ny \quad \text{et} \quad -Q = \delta y + mx.$$

fferentiationem vero obtinemus

$$dx(mx + \delta y) + dy(ny + \delta x) = 0$$

$dy = 0$ ideoque $\frac{dy}{Q} = \frac{dx}{P}$, unde aequatio assumta huic aequa-

$$\int \frac{dy}{Q} = \int \frac{dx}{P}$$

m X et Y functiones ipsarum x et y singulatim ac ponatur

$$\int \frac{Xdx}{P} - \int \frac{Ydy}{Q} = V,$$

uantitas algebraica, eritque

$$\frac{(X - Y)dx}{P} = dV = \frac{(X - Y)dx}{\delta x + ny}.$$

31. Posito $xy = u$, ut sit $dy = \frac{du}{x} - \frac{ydx}{x}$, erit

$$dx(mxx - nyy) + du(ny + \delta x) = 0,$$

unde, cum fiat $\frac{dx}{\delta x + ny} = \frac{-du}{mxx - nyy}$, erit

$$dV = \frac{-du(X - Y)}{mxx - nyy}$$

hincque non difficulter casus integrabiles eliciuntur.

32. Sit enim primo $X = mxx$ et $Y = nyy$; erit

$$dV = -du \quad \text{et} \quad V = -u = -xy.$$

Hinc relatio inter x et y assumta satisfacit huic aequationi in

$$\int\frac{mxxdx}{P} - \int\frac{nyydy}{Q} = \text{Const.} - xy.$$

33. Sit secundo $X = mmx^4$ et $Y = nny^4$; erit

$$dV = -du(mxx + nyy) = +du(\alpha + 2\delta u),$$

unde fit

$$V = u(\alpha + \delta u) = xy(\alpha + \delta xy).$$

Ergo huic aequationi integrali

$$\int\frac{mmx^4dx}{P} - \int\frac{nny^4dy}{Q} = \text{Const.} + xy(\alpha + \delta xy)$$

satisfacit relatio assumta inter x et y.

34. His igitur colligendis relatio inter x et y assumta s
aequationi integrali

$$\int\frac{dx(\mathfrak{A} + \mathfrak{B}mxx + \mathfrak{C}m^2x^4)}{\sqrt{((\delta\delta - mn)xx - \alpha n)}} - \int\frac{dy(\mathfrak{A} + \mathfrak{B}nyy + \mathfrak{C}n^2y^4)}{\sqrt{((\delta\delta - mn)yy - \alpha m)}}$$
$$= \text{Const.} - \mathfrak{B}xy + \mathfrak{C}xy(\alpha + \delta xy).$$

mus ad faciliorem applicationem

$$\delta\delta - mn = Cp, \quad \alpha n = -Ap \quad \text{et} \quad \alpha m = -Bp,$$

$$P = \sqrt{p(A + Cxx)} \quad \text{et} \quad Q = \sqrt{p(B + Cyy)};$$

Sit ergo $m = B$ et $n = A$; erit

$$\alpha = -p \quad \text{et} \quad \delta = \sqrt{(AB + Cp)}.$$

Ckk, ut sit $\alpha = -Ckk$, et aequatio relationem inter x et y

$$0 = -Ckk + Bxx + Ayy + 2xy\sqrt{(AB + CCkk)}.$$

ob rem valores ipsius x et y hinc erunt

$$y = \frac{-x\sqrt{(AB + CCkk)} + k\sqrt{C(A + Cxx)}}{A},$$

$$x = \frac{-y\sqrt{(AB + CCkk)} - k\sqrt{C(B + Cyy)}}{B}$$

$$P = k\sqrt{C(A + Cxx)} \quad \text{et} \quad Q = k\sqrt{C(B + Cyy)}.$$

itur valores conveniunt huic aequationi integrali

$$\int \frac{dx(\mathfrak{A} + \mathfrak{B}Bxx + \mathfrak{C}B^2x^4)}{\sqrt{(A + Cxx)}} = \int \frac{dy(\mathfrak{A} + \mathfrak{B}Ayy + \mathfrak{C}A^2y^4)}{\sqrt{(B + Cyy)}}$$

$$\text{st.} - \mathfrak{B}kxy\sqrt{C} + \mathfrak{C}kxy(-Ckk + xy\sqrt{(AB + C^2kk)})\sqrt{C}.$$

ur $B = \frac{CE}{F}$, quae aequatio latius patere videatur, atque con-
itis prodibit ista aequatio integralis

$$\frac{\mathfrak{A} + \frac{C}{A}\mathfrak{B}xx + \frac{CC}{AA}\mathfrak{C}x^4)\sqrt{C}}{\sqrt{(A + Cxx)}} = \int \frac{dy\left(\mathfrak{A} + \frac{F}{E}\mathfrak{B}yy + \frac{FF}{EE}\mathfrak{C}y^4\right)\sqrt{F}}{\sqrt{(E + Fyy)}}$$

$$- \frac{CF}{AE}\mathfrak{B}kxy - \frac{CCFF}{AAEE}\mathfrak{C}k^3xy + \frac{CCFF}{AAEE}\mathfrak{C}kxxyy\sqrt{\left(\frac{AE}{CF} + kk\right)},$$

$$kk = \frac{E}{F}xx + \frac{A}{C}yy + 2xy\sqrt{\left(\frac{AE}{CF} + kk\right.}$$

39. Hae formulae ratione signorum utcunque transm
enim in formulis integralibus nihil mutando tam k qu
lubitu vel affirmative vel negative accipi possunt, du
ratio ubique observetur. Deinde etiam tam $\sqrt{C}$ quan
potest; illo autem casu quoque $\sqrt{\left(\frac{A}{C} + xx\right)}$, quippe
$\sqrt{\left(\frac{E}{F} + yy\right)}$ negative est accipiendum.

40. Denique patet, si C sit quantitas positiva, tum
positivam esse oportere, quia alioquin altera formula i
naria. Sin autem C sit quantitas negativa, tum etiar
est; et quo hoc casu imaginaria se destruant, pro
accipienda erit, quo k et k^3 fiant quoque imaginariae.

41. Hoc ergo casu sequens habebitur aequatio inte

$$\int \frac{dx\left(\mathfrak{A} + \frac{C}{A}\mathfrak{B}xx + \frac{CC}{AA}\mathfrak{C}x^4\right)\sqrt{C}}{\sqrt{(A - Cxx)}} - \int \frac{dy\left(\mathfrak{A} + \frac{F}{E}\mathfrak{B}yy\right.}{\sqrt{(E -}}$$

$$= \text{Const.} + \frac{CF}{AE}\mathfrak{B}kxy + \frac{CCFF}{AAEE}\mathfrak{C}k^3xy + \frac{CCFF}{AAEE}\mathfrak{C}kx$$

cui satisfaciunt isti valores

$$\frac{Ay}{C} = x\sqrt{\left(\frac{AE}{CF} - kk\right)} - k\sqrt{\left(\frac{A}{C} - xx\right.}$$

$$\frac{Ex}{F} = y\sqrt{\left(\frac{AE}{CF} - kk\right)} + k\sqrt{\left(\frac{E}{F} - yy\right.}$$

one oriundi

$$kk = \frac{E}{F}xx + \frac{A}{C}yy - 2xy\sqrt{\left(\frac{AE}{CF} - kk\right)}.$$

ormulae etiam eas, quae ex hypothesi prima sunt erutae, in
ur, ponendo scilicet $E = A$ et $F = C$; quin etiam formulae
thesis his non latius patent. Si enim in relatione secundo
pro $x + \frac{\beta}{\gamma + \delta}$ et $y + \frac{\beta}{\gamma + \delta}$ scribatur x et y, aequatio omnino
oritur similique modo, si hanc relationem constituere velimus

$$0 = \alpha + 2bx + 2\beta y + \gamma xx + cyy + 2\delta xy,$$

tionem tertiam reduceretur, unde eius evolutionem praetermitto.

cuum nunc est ex his formulis infinitas comparationes institui
ntitates transcendentes tam ratione spatiorum quam arcuum.
a quadratura circuli pendent vel a logarithmis. Etsi autem
nes etiam vulgari calculo institui possunt, tamen non inutile
quemadmodum eaedem multo facilius ex his formulis derivari
eo magis notatu dignum videtur, cum hic neque naturae cir-
rithmorum ratio peculiaris habeatur. Ex quo facilius intelli-
modum haec methodus etiam pari successu ad eiusmodi for-
s se extendat, quae neque ad circuli neque hyperbolae quadra-
possunt.

DE COMPARATIONE ARCUUM CIRCULARIUM

lius circuli seu sinus totus $= 1$ ac posito sinu quocunque $= z$
spondens $= \Pi . z$, sumto Π pro nota eius functionis, qua pen-
suo sinu denotatur. Erit ergo, uti constat,

$$\Pi . z = \int \frac{dz}{\sqrt{(1 - zz)}};$$

las integrales § 41 erutas huc transferamus, poni oportet

$$= E = C = F = 1, \quad \mathfrak{A} = 1, \quad \mathfrak{B} = 0 \quad \text{et} \quad \mathfrak{C} = 0.$$

45. Ex his autem valoribus emerget haec aequatio integ

$$\int\frac{dx}{\sqrt{(1-xx)}}-\int\frac{dy}{\sqrt{(1-yy)}}=\text{Const.},$$

cui satisfacere inventae sunt hae formulae

$$y=x\sqrt{(1-kk)}-k\sqrt{(1-xx)},$$

$$x=y\sqrt{(1-kk)}+k\sqrt{(1-yy)},$$

quae oriuntur ex hac aequatione

$$kk=xx+yy-2xy\sqrt{(1-kk)}.$$

46. Per has igitur determinationes satisfit huic aequat

$$\Pi.x-\Pi.y=\text{Const.},$$

in qua constans ita determinabitur: ponatur $y=0$ eritque casu prodit $\Pi.k-\Pi.0=\text{Const.}$ seu ob $\Pi.0=0$ erit C arcui, cuius sinus $=k$. Hinc generatim habebimus

$$\Pi.x-\Pi.y=\Pi.k.$$

47. Hinc ergo statim arcuum tam additio quam subtrac enim duo habeantur arcus $\Pi.k$ et $\Pi.y$, quarum sinus summae arcuum sinus ponatur $=x$, ut sit $\Pi.x=\Pi.k+\Pi$

$$x=y\sqrt{(1-kk)}+k\sqrt{(1-yy)}.$$

Porro si maioris arcus sinus sit $=x$, minoris $=k$ sinu ponatur $=y$, ut sit $\Pi.y=\Pi.x-\Pi.k$, erit

$$y=x\sqrt{(1-kk)}-k\sqrt{(1-xx)},$$

uti ex elementis est manifestum.

48. Perspicuum etiam est, quemadmodum hinc arcuum deduci oporteat. Posito enim $y=k$, ut sit

$$x=2k\sqrt{(1-kk)},$$

erit

$$\Pi.x=2\Pi.k.$$

ic pro x inventus loco y substituatur, in formula

$$x = y\sqrt{(1-kk)} + k\sqrt{(1-yy)}$$

$I.\,k$ prodibit

$$\Pi.\,x = 3\Pi.\,k.$$

nere autem, si sit y sinus arcus nk seu $\Pi.\,y = n\Pi.\,k$ et $\sqrt{(1-yy)}$ cus nk, uti $\sqrt{(1-kk)}$ denotat cosinum arcus k, atque ponatur $k) + k\sqrt{(1-yy)}$, erit

$$\Pi.\,x = (n+1)\Pi.\,k.$$

cuiusvis multipli arcus k reperietur sinus multipli unitate

autem haec facilius expediri queant, valorem quoque ipsius cosinus sse conveniet; quem in finem, cum ex formula prima sit

$$k\sqrt{(1-xx)} = x\sqrt{(1-kk)} - y,$$

ic valor ipsius x ex altera formula; erit

$$k\sqrt{(1-xx)} = y(1-kk) + k\sqrt{(1-kk)(1-yy)} - y$$

$$\sqrt{(1-xx)} = \sqrt{(1-kk)(1-yy)} - ky$$

do erit

$$\sqrt{(1-yy)} = \sqrt{(1-kk)(1-xx)} + kx.$$

ntis ergo valoribus tam pro x quam pro $\sqrt{(1-xx)}$ multiplicetur productum ad hunc addatur eritque

$$+ \lambda x = \sqrt{(1-kk)(1-yy)} - ky + \lambda y\sqrt{(1-kk)} + \lambda k\sqrt{(1-yy)}$$

$$x) + \lambda x = \left(\sqrt{(1-kk)} + \lambda k\right)\sqrt{(1-yy)} + y\left(\lambda\sqrt{(1-kk)} - k\right).$$

factores similes reddantur, necesse est, ut sit $\lambda = \sqrt{-1}$, eritque

$$xx) + x\sqrt{-1} = \left(\sqrt{(1-kk)} + k\sqrt{-1}\right)\left(\sqrt{(1-yy)} + y\sqrt{-1}\right).$$

52. Hanc ergo formulam loco superioris adhibendo s
$\Pi. x = 2\Pi. k$, ob $y = k$ esse oportere

$$\sqrt{(1-xx)} + x\sqrt{-1} = \left(\sqrt{(1-kk)} + k\sqrt{-1}\right.$$

Ac si hic valor pro x inventus loco y substituatur, ut
prodibit

$$\sqrt{(1-xx)} + x\sqrt{-1} = \left(\sqrt{(1-kk)} + k\sqrt{-}\right.$$

pro $\Pi. x = 3\Pi. k$, unde in genere colligitur, ut sit $\Pi. x =$

$$\sqrt{(1-xx)} + x\sqrt{-1} = \left(\sqrt{(1-kk)} + k\sqrt{-1}\right.$$

53. Quia porro $\sqrt{-1}$ ob suam naturam tam negativ
accipere licet, erit quoque pro eadem arcus multiplicatione

$$\sqrt{(1-xx)} - x\sqrt{-1} = \left(\sqrt{(1-kk)} - k\sqrt{-1}\right.$$

ideoque vel

$$\sqrt{(1-xx)} = \frac{\left(\sqrt{(1-kk)} + k\sqrt{-1}\right)^n + \left(\sqrt{(1-kk)} - k\right.}{2}$$

vel

$$x = \frac{\left(\sqrt{(1-kk)} + k\sqrt{-1}\right)^n - \left(\sqrt{(1-kk)} - k\sqrt{-}\right.}{2\sqrt{-1}}$$

quae formulae quoque valent pro valoribus fractis exponen

II. DE COMPARATIONE ARCUUM PARABOLI

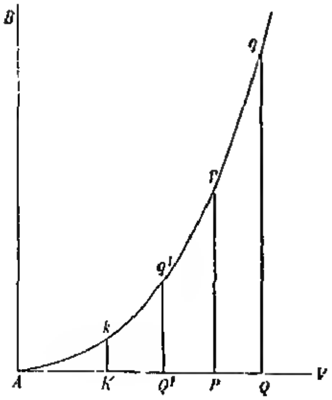


Fig. 1.

54. Sit AB (Fig. 1) axis
bolae, quem tangat recta in
qua capiantur abscissae; pos
latere recto $= 2$ sit abscissa
erit applicata $Pp = \frac{1}{2}zz$, ex q
huic abscissae respondens erit
qui cum sit functio ipsius z,
ita ut $\Pi. z$ significet arcum pa
convenientem seu sit

$$\Pi. z = \int dz\sqrt{(1-}$$

rationalitate in denominatorem translata erit

$$\Pi . z = \int \frac{dz(1+zz)}{\sqrt{(1+zz)}}.$$

go formam ut formulae integrales § 38 revocentur, erit

$$= E = 1, \quad C = F = 1, \quad \mathfrak{A} = 1 \quad \text{et} \quad \mathfrak{B} = 1 \quad \text{atque} \quad \mathfrak{C} = 0.$$

tio illa integralis in hanc abit formam

$$\int \frac{dx(1+xx)}{\sqrt{(1+xx)}} - \int \frac{dy(1+yy)}{\sqrt{(1+yy)}} = \text{Const.} + kxy,$$

iunt hi valores

$$k\sqrt{(1+xx)} + x\sqrt{(1+kk)} \quad \text{et} \quad x = k\sqrt{(1+yy)} + y\sqrt{(1+kk)}$$

k quam $\sqrt{(1+kk)}$ negativis.

ac igitur inter x et y relatione subsistente pro arcubus para-

$$\Pi . x - \Pi . y = \text{Const.} + kxy;$$

onstantem determinandam ponatur $y = 0$, et quia tum fit $x = k$, Const. Quocirca habebitur

$$\Pi . x - \Pi . y = \Pi . k + kxy.$$

igitur haec aequatio locum habeat, relatio inter ternas abscissas k, modi erit

$$\sqrt{(1+yy)} + y\sqrt{(1+kk)} \quad \text{seu} \quad y = x\sqrt{(1+kk)} - k\sqrt{(1+xx)},$$

erea oriuntur istae determinationes

$$= \sqrt{(1+kk)(1+yy)} + ky \quad \text{et} \quad \sqrt{(1+yy)} = \sqrt{(1+kk)(1+xx)} - kx,$$

porro elicitur

$$x + \sqrt{(1+xx)} = (k + \sqrt{(1+kk)})(y + \sqrt{(1+yy)}).$$

ut sit

$$q = k\sqrt{(1+pp)} + p\sqrt{(1+kk)} \quad \text{et} \quad p = q\sqrt{(1+$$

seu

$$q + \sqrt{(1+qq)} = (k + \sqrt{(1+kk)})(p + \sqrt{(}$$

erit

$$\Pi.q - \Pi.p = \Pi.k + kpq.$$

Ideoque hanc aequationem ab illa subtrahendo habebi

$$(\Pi.x - \Pi.y) - (\Pi.q - \Pi.p) = k(xy$$

59. Pro hoc igitur casu erit

$$\frac{x+\sqrt{(1+xx)}}{y+\sqrt{(1+yy)}} = \frac{q+\sqrt{(1+qq)}}{p+\sqrt{(1+pp)}},$$

unde relatio inter p, q, x et y sine k obtinetur. Eri

$$k = x\sqrt{(1+yy)} - y\sqrt{(1+xx)} = q\sqrt{(1+pp)}$$

et

$$\sqrt{(1+kk)} = \sqrt{(1+xx)(1+yy)} - xy = \sqrt{(1+}$$

60. Iam ob

$$\frac{1}{p+\sqrt{(1+pp)}} = \sqrt{(1+pp)} - p$$

erit

$$\sqrt{(1+xx)} + x = (\sqrt{(1+yy)} + y)(\sqrt{(1+qq)} + q$$

unde reperitur

$$x = y\sqrt{(1+pp)(1+qq)} + q\sqrt{(1+pp)(1+yy)} - p\sqrt{}$$

Quare erit

$$(\Pi.x - \Pi.y) - (\Pi.q - \Pi.p)$$

$$= (q\sqrt{(1+pp)} - p\sqrt{(1+qq)})(y\sqrt{(1+pp)} - p\sqrt{(1+yy)})$$

PROBLEMA 1

rcu parabolae quocunque Ak (Fig. 1, p. 128) *in vertice A terminato ue puncto p arcum abscindere pq, qui arcum illum Ak superet aice assignabili.*

SOLUTIO

abolae parametro $= 2$ sit k abscissa arcui Ak conveniens, ab-punctis p et q respondentes sint $AP = y$ et $AQ = x$ eritque

$$\text{Arc. } pq = \Pi.\, x - \Pi.\, y \quad \text{et} \quad \text{Arc. } Ak = \Pi.\, k;$$

sit abscissa $AP = y$, si capiatur altera

$$AQ = x = y\sqrt{(1 + kk)} + k\sqrt{(1 + yy)},$$

$$\Pi.\, x - \Pi.\, y = \Pi.\, k + kxy$$

$$\text{Arc. } pq = \text{Arc. } Ak + kxy.$$

arcus pq, qui in dato puncto p terminatur, arcum Ak quan-o assignabili kxy.

am a puncto p antrorsum abscindi arcus pq^1, qui pariter titate geometrica superet; ad hoc ponatur $AP = x$ et $AQ^1 = y$ $+ kk) - k\sqrt{(1 + xx)}$; et cum sit $\text{Arc. } pq^1 = \Pi.\, x - \Pi.\, y$, erit

$$\text{Arc. } pq^1 = \text{Arc. } Ak + kxy.$$

solutio ita coniungetur, ut posita abscissa data $AP = p$

$$+ kk) + k\sqrt{(1 + pp)} \quad \text{et} \quad AQ^1 = p\sqrt{(1 + kk)} - k\sqrt{(1 + pp)},$$

$$\text{Arc. } pq = \text{Arc. } Ak + kp \cdot AQ,$$

$$\text{Arc. } pq^1 = \text{Arc. } Ak + kp \cdot AQ^1,$$

modo problemati est satisfactum.

COROLLARIUM 1

autem nequit, ut excessus kxy, quo arcus pq arcum Ak cat; deberet enim esse vel $x = 0$ vel $y = 0$. At casu $x = 0$

Ak similis capiendus; altero autem casu, quo $y = 0$, fie
in arcum Ak abiret; unde arcui Ak geometrice in pa
alius arcus ipsi aequalis, qui ipsi non simul futurus si

COROLLARIUM 2

63. Vicissim ergo dato arcu quocunque pq in para
arcus abscindi poterit Ak, qui ab illo deficiat quantit
enim nunc datae sint abscissae $AP = y$ et $AQ = x$, er

$$AK = k = x\sqrt{(1+yy)} - y\sqrt{(1+xx$$

qua inventa erit Arc. pq — Arc. $Ak = kxy$.

COROLLARIUM 3

64. Quin etiam puncto p pro incognito habito, pr
arcus pq assignari poterit, qui illum superet quanti
Habebimus ergo has duas aequationes

$$kxy = C \quad \text{et} \quad xx + yy = kk + 2xy\sqrt{(1$$

seu

$$xx + yy = kk + \frac{2C}{k}\sqrt{(1+kk)};$$

ergo

$$x + y = \sqrt{\left(kk + \frac{2C}{k} + \frac{2C}{k}\sqrt{(1+kk}\right.$$

$$x - y = \sqrt{\left(kk - \frac{2C}{k} + \frac{2C}{k}\sqrt{(1+kk}\right.$$

Seu sint x et y binae radices huius aequationis quadra

$$zz - Pz + Q = 0;$$

erit

$$Q = \frac{C}{k} \quad \text{et} \quad P = \sqrt{\left(kk + \frac{2C}{k} + \frac{2C}{k}\sqrt{(1\right.}$$

unde

$$z = \frac{1}{2}\sqrt{\left(kk + \frac{2C}{k} + \frac{2C}{k}\sqrt{(1+kk)}\right)} \pm \frac{1}{2}\sqrt{\left(kk - \frac{2C}{k}\right.}$$

COROLLARIUM 4

Quantacunque sit haec quantitas C, modo sit affirmativa, semper pro x et y valores reales iique affirmativi. At si sit $C=0$, fiet $y=0$. Quin etiam poni potest C negativum, quo casu y reperitur egativum et arcus quaesitus utrinque circa verticem A erit dispo- rum si sit $C=-D$, necesse est, ut sit

$$D < \frac{k^3}{2(1+\sqrt{(1+kk)})} \quad \text{seu} \quad D < \frac{1}{2}k\left(\sqrt{(1+kk)}-1\right);$$

esset maius, utraque abscissa fieret imaginaria.

COROLLARIUM 5

Casu autem

$$D = -C = \frac{1}{2}k\left(\sqrt{(1+kk)}-1\right) \quad \text{erit} \quad zz = \frac{D}{k}$$

$$= +\sqrt{\frac{1}{2}\left(\sqrt{(1+kk)}-1\right)} \quad \text{et} \quad y = -\sqrt{\frac{1}{2}\left(\sqrt{(1+kk)}-1\right)};$$

su orietur arcus utrinque a vertice aeque extensus, cuius defectus Mk est minimus omnium, qui quidem geometrice construi possunt.

PROBLEMA 2

Dato arcu parabolae quocunque ef (Fig. 2) *a dato eius puncto quocunque bscindere arcum pq, ita ut arcuum ef et pq differentia geometrice possit*

SOLUTIO

o parabolae latere recto $=2$ tangat recta bolam in vertice A, a qua capiantur ab- uae sint $AE=e$, $AF=f$, $AP=p$ et quarum tres priores e, f, p sunt datae, q ita accipiatur, ut sit per § 59

$$\frac{q+\sqrt{(1+qq)}}{p+\sqrt{(1+pp)}} = \frac{f+\sqrt{(1+ff)}}{e+\sqrt{(1+ee)}}.$$

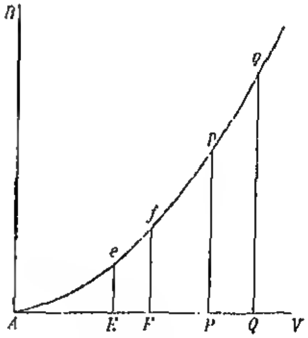


Fig. 2.

Tum vero fit

$$k = f\sqrt{(1+ee)} - e\sqrt{(1+ff)}$$

scribendo e et f pro y et x eritque

$$(\Pi.q - \Pi.p) - (\Pi.f - \Pi.e) = k(pq - e$$

Ideoque habebitur

$$\text{Arc. } pq - \text{Arc. } ef = k(pq - ef).$$

Hinc etiam apparet, si punctum q fuerit datum, ex fo
modo punctum p antrorsum procedendo definiri posse, u
prodeat geometrice assignabilis.

COROLLARIUM 1

68. Ex reductione § 60 facta patet esse

$$pq - ef = \left(p\sqrt{(1+ee)} - e\sqrt{(1+pp)}\right)\left(p\sqrt{(1+ff)}\right.$$

sicque sumta abscissa q ex aequatione

$$\frac{q+\sqrt{(1+qq)}}{p+\sqrt{(1+pp)}} = \frac{f+\sqrt{(1+ff)}}{e+\sqrt{(1+ee)}}$$

erit

$$\text{Arc. } pq - \text{Arc. } ef$$

$$= \left(f\sqrt{(1+ee)} - e\sqrt{(1+ff)}\right)\left(p\sqrt{(1+ee)} - e\sqrt{(1+pp)}\right)\left(p\sqrt{(1\right.$$

COROLLARIUM 2

69. Si velimus punctum p ita accipere, ut arcuum
seu fiat Arc. $pq =$ Arc. ef, oportet esse

$$\text{vel } p\sqrt{(1+ee)} - e\sqrt{(1+pp)} = 0 \quad \text{vel } p\sqrt{(1+ff)} +$$

Priori casu fit $p = \pm e$, posteriori $p = \pm f$, utroque aute
cum arcu ef congruit vel eius fit similis in altero paral
ita ut geometrice duo arcus aequales exhiberi nequeant,
futuri sint similes.

COROLLARIUM 3

it $k = f\sqrt{(1+ee)} - e\sqrt{(1+ff)}$, erit

$$\sqrt{(1+kk)} = \sqrt{(1+ee)(1+ff)} - ef;$$

$$= f\sqrt{(1+ff)} + 2eef\sqrt{(1+ff)} - 2eff\sqrt{(1+ee)} - e\sqrt{(1+ee)}$$

$$= f\sqrt{(1+ff)} - e\sqrt{(1+ee)} - 2ef\big(f\sqrt{(1+ee)} - e\sqrt{(1+ff)}\big)$$

$$k\sqrt{(1+kk)} = f\sqrt{(1+ff)} - e\sqrt{(1+ee)} - 2efk.$$

bitur

$$f = \frac{1}{2} f\sqrt{(1+ff)} - \frac{1}{2} e\sqrt{(1+ee)} - \frac{1}{2} k\sqrt{(1+kk)}.$$

COROLLARIUM 4

gitur k simili quoque modo pendet a p et q, erit etiam

$$q = \frac{1}{2} q\sqrt{(1+qq)} - \frac{1}{2} p\sqrt{(1+pp)} - \frac{1}{2} k\sqrt{(1+kk)}.$$

um differentia sit $= kpq - kef$, si quatuor parabolae puncta se invicem pendent, ut sit

$$\frac{q+\sqrt{(1+qq)}}{p+\sqrt{(1+pp)}} = \frac{f+\sqrt{(1+ff)}}{e+\sqrt{(1+ee)}},$$

$$f = \frac{1}{2} q\sqrt{(1+qq)} - \frac{1}{2} p\sqrt{(1+pp)} - \frac{1}{2} f\sqrt{(1+ff)} + \frac{1}{2} e\sqrt{(1+ee)},$$

ob functiones quantitatum p, q, e, f a se invicem separatas a.

COROLLARIUM 5

o inter e, f, p, q etiam ita exprimi potest, ut sit

$$q) + q = \big(\sqrt{(1+ee)} - e\big)\big(\sqrt{(1+ff)} + f\big)\big(\sqrt{(1+pp)} + p\big);$$

$$\frac{1}{\sqrt{(1+qq)}+q} = \sqrt{(1+qq)} - q$$

erit

$$\sqrt{(1+qq)} - q = \left(\sqrt{(1+ee)}+e\right)\left(\sqrt{(1+ff)}-f\right)\left(\right.$$

unde datis e, f et p facile valor tam pro q quam pro

COROLLARIUM 6

73. Ex formula corollario 1 data apparet arcum
fore arcu ef, si punctum p a vertice parabolae A
quam punctum e, contra autem arcum pq proditurum
quidem sit $p = 0$, erit

$$\text{Arc. } ef - \text{Arc. } pq = ef\left(f\sqrt{(1+ee)} - e\sqrt{(1}\right.$$

minimus autem omnium arcus pq evadet, si capiatur

$$p = -\sqrt{\tfrac{1}{2}}\left(\sqrt{(1+ee)(1+ff)} - ef\right.$$

et

$$q = +\sqrt{\tfrac{1}{2}}\left(\sqrt{(1+ee)(1+ff)} - ef\right.$$

tumque erit

$$\text{Arc. } ef - \text{Arc. } pq = \tfrac{1}{2}(e+f)\left(\sqrt{(1+ff)} -\right.$$

Arcusque pq utrinque aeque circa verticem A erit dis

PROBLEMA 3

74. *Dato arcu parabolae ef* (Fig. 3, p. 137) *a puncto*
pz, qui superet datum multiplum arcus ef quantitate geo

SOLUTIO

Posito parabolae latere recto $= 2$ sint in ver
datae $AE = e$, $AF = f$ et $AP = p$; tum capiantur abs

t et ultima sit $AZ = z$; quae ita determinentur, ut sit primo

$$\frac{q+\sqrt{(1+qq)}}{p+\sqrt{(1+pp)}} = \frac{f+\sqrt{(1+ff)}}{e+\sqrt{(1+ee)}},$$

$$f = \frac{1}{2} q\sqrt{(1+qq)} - \frac{1}{2} p\sqrt{(1+pp)} - \frac{1}{2} f\sqrt{(1+ff)} + \frac{1}{2} e\sqrt{(1+ee)}.$$

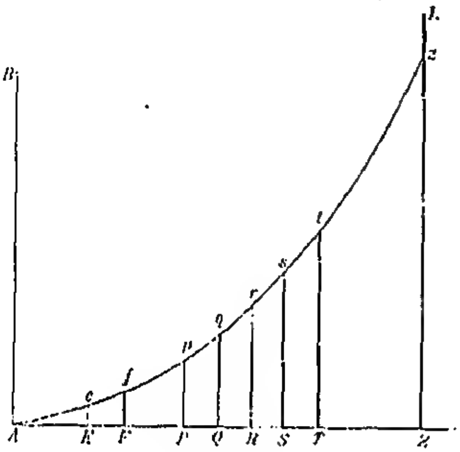


Fig. 8.

cto q simili modo definiatur punctum r, ut sit

$$\frac{\ldots rr)}{\ldots qq)} = \frac{f+\sqrt{(1+ff)}}{e+\sqrt{(1+ee)}} \quad \text{seu} \quad \frac{r+\sqrt{(1+rr)}}{p+\sqrt{(1+pp)}} = \left(\frac{f+\sqrt{(1+ff)}}{e+\sqrt{(1+ee)}}\right)^2,$$

$$= \frac{1}{2} r\sqrt{(1+rr)} - \frac{1}{2} q\sqrt{(1+qq)} - \frac{1}{2} f\sqrt{(1+ff)} + \frac{1}{2} e\sqrt{(1+ee)},$$

ad illam addita prodibit

$$zf = \frac{1}{2} r\sqrt{(1+rr)} - \frac{1}{2} p\sqrt{(1+pp)} - \frac{2}{2} f\sqrt{(1+ff)} + \frac{2}{2} e\sqrt{(1+ee)}.$$

to r capiatur punctum s, ut sit

$$\frac{\ldots ss)}{\ldots rr)} = \frac{f+\sqrt{(1+ff)}}{e+\sqrt{(1+ee)}} \quad \text{seu} \quad \frac{s+\sqrt{(1+ss)}}{p+\sqrt{(1+pp)}} = \left(\frac{f+\sqrt{(1+ff)}}{e+\sqrt{(1+ee)}}\right)^3,$$

$$= \frac{1}{2} s\sqrt{(1+ss)} - \frac{1}{2} r\sqrt{(1+rr)} - \frac{1}{2} f\sqrt{(1+ff)} + \frac{1}{2} e\sqrt{(1+ee)},$$

$$\text{Arc.}\,ps - 3\,\text{Arc.}\,ef = \frac{1}{2}s\sqrt{(1+ss)} - \frac{1}{2}p\sqrt{(1+pp)} - \frac{3}{2}f\sqrt{(1+ff)} +$$

Atque hoc modo si ulterius progrediamur sitque z punctum post
operationes inventum, erit

$$\frac{z+\sqrt{(1+zz)}}{p+\sqrt{(1+pp)}} = \left(\frac{f+\sqrt{(1+ff)}}{e+\sqrt{(1+ee)}}\right)^n,$$

unde immediate punctum z reperietur, ita ut sit

$$\text{Arc.}\,pz - n\,\text{Arc.}\,ef = \frac{1}{2}z\sqrt{(1+zz)} - \frac{1}{2}p\sqrt{(1+pp)} - \frac{n}{2}f\sqrt{(1+ff)} +$$

sicque arcus pz est inventus a dato puncto p abscissus, q
n vicibus sumtum superat quantitate geometrica.

COROLLARIUM 1

75. Quodcunque ergo multiplum arcus ef proponatur, cuius
ponens sit numerus n, sive is sit integer sive fractus, a dato pu
abscindi poterit arcus pz, qui hoc multiplum excedat quantita
assignabili; erit enim

$$\sqrt{(1+zz)} + z = (\sqrt{(1+pp)}+p)(\sqrt{(1+ff)}+f)^n(\sqrt{(1+ee)}$$

et

$$\sqrt{(1+zz)} - z = (\sqrt{(1+pp)}-p)(\sqrt{(1+ff)}-f)^n(\sqrt{(1+ee)}$$

COROLLARIUM 2

76. Quodsi ergo ad abbreviandum ponatur

$$\sqrt{(1+ee)}+e = E, \quad \sqrt{(1+ff)}+f = F, \quad \sqrt{(1+pp)}+p$$

erit

$$\sqrt{(1+zz)}+z = \frac{PF^n}{E^n} \quad \text{et} \quad \sqrt{(1+zz)}-z = \frac{E^n}{PF^n},$$

unde oritur

$$\sqrt{(1+zz)} = \frac{P^2F^{2n}+E^{2n}}{2PE^nF^n} \quad \text{et} \quad z = \frac{P^2F^{2n}-E^{2n}}{2PE^nF^n}.$$

COROLLARIUM 3

rgo fiet

$$\frac{1}{2} z \sqrt{(1+zz)} = \frac{P^4 F^{4n} - E^{4n}}{8 P^2 E^{2n} F^{2n}}.$$

i modo est

$$= \frac{E^4-1}{8EE}, \quad \frac{1}{2} f \sqrt{(1+ff)} = \frac{F^4-1}{8FF} \quad \text{et} \quad \frac{1}{2} p \sqrt{(1+pp)} = \frac{P^4-1}{8PP},$$

$$- n \,\text{Arc.}\, cf = \frac{P^4 F^{4n} - E^{4n}}{8 P^2 E^{2n} F^{2n}} - \frac{P^4-1}{8PP} - \frac{n(F^4-1)}{8FF} + \frac{n(E^4-1)}{8EE}.$$

COROLLARIUM 4

us expressionis partes binae in unum congregentur, reperietur
geometrica

$$\text{Arc.}\, cf = \frac{(F^{2n} - E^{2n})(P^4 F^{2n} + E^{2n})}{8 P^2 E^{2n} F^{2n}} - \frac{n(FF-EE)(EEFF+1)}{8EEFF}.$$

COROLLARIUM 5

dmodum hic ex puncto dato p alterum punctum z deter-
vicissim, si punctum z pro dato accipiatur, antrorsum pro-
i modo punctum p ex eadem aequatione reperietur, ita ut
arcum cf n vicibus sumtum quantitate geometrica assignabili.

PROBLEMA 4

*in parabola arcu quocunque cf invenire alium arcum pz, qui se
in data ratione $n:1$, ita ut sit $\text{Arc.}\, pz = n \,\text{Arc.}\, cf$.*

SOLUTIO

sdem denominationibus, quibus in probl. praecedenti eiusque
mus, quoniam fieri debet

$$\text{Arc.}\, pz - n \,\text{Arc.}\, cf = 0,$$

quantitas illa algebraica, cui haec arcuum differentia ae
nihilum abire debet. Habebimus ergo ex corollario 4 h

$$F^{2n}P^4 + E^{2n} = \frac{n E^{2n-2}F^{2n-2}(FF-EE)(EEF}{F^{2n}-E^{2n}}$$

Ponamus brevitatis gratia $\frac{F}{E} = C$ eritque

$$C^{2n}P^4 + 1 = \frac{n C^{2n-2}(CC-1)(CCE^4+1)}{(C^{2n}-1)EE}$$

unde fit

$$C^n P^2 = \frac{n C^{n-2}(CC-1)(CCE^4+1)}{2(C^{2n}-1)EE} - \sqrt{\Big(\frac{nn C^{2n-4}(CC-1}{4(C^{2n}-}}$$

ideoque

$$P = \sqrt{\Big(\frac{n(CC-1)(CCE^4+1)}{2(C^{2n}-1)CCEE}} - \sqrt{\Big(\frac{nn(CC-1)^2(CC}{4(C^{2n}-1)^2C}}$$

sive

$$P = \sqrt{\Big(\frac{n(CC-1)(CCE^4+1)}{4(C^{2n}-1)CCEE} + \frac{1}{2C^n}\Big)} - \sqrt{\Big(\frac{n(CC-1)(}{4(C^{2n}-1}}$$

Deinde si pari modo ponatur $\sqrt{(1+zz)} + z = Z$, erit Z
autem quantitatibus P et Z ita eliciuntur ipsae p et z

$$p = \frac{PP-1}{2P} \quad \text{et} \quad z = \frac{ZZ-1}{2Z}.$$

Restituto autem pro C valore $\frac{F}{E}$ si ponamus

$$\sqrt{\Big(\frac{n(FF-EE)(EEFF+1)}{4EEFF(F^{2n}-E^{2n})} + \frac{1}{2E^nF^n}\Big)} =$$

$$\sqrt{\Big(\frac{n(FF-EE)(EEFF+1)}{4EEFF(F^{2n}-E^{2n})} - \frac{1}{2E^nF^n}\Big)} =$$

reperietur

$$P = E^n(M-N) \quad \text{et} \quad \frac{1}{P} = F^n(M+$$

$$Z = F^n(M-N) \quad \text{et} \quad \frac{1}{Z} = E^n(M+$$

unde concluduntur ipsae abscissae

$$p = -\tfrac{1}{2}M(F^n - E^n) - \tfrac{1}{2}N(F^n + $$

$$z = +\tfrac{1}{2}M(F^n - E^n) - \tfrac{1}{2}N(F^n + $$

M et N tam affirmative quam negative accipere liceat, capiatur ut punctum z in istum parabolae ramum incidat, in quo est que

$$p = \frac{1}{2} N(F^n + E^n) - \frac{1}{2} M(F^n - E^n),$$

$$z = \frac{1}{2} N(F^n + E^n) + \frac{1}{2} M(F^n - E^n).$$

rmulis si definiantur puncta p et z, erit

$$\text{Arc. } pz = n \text{ Arc. } ef.$$

COROLLARIUM 1

ae ergo abscissae $AP = p$ et $AZ = z$ ita sunt comparatae, ut sit

$$z + p = N(F^n + E^n) \quad \text{et} \quad z - p = M(F^n - E^n).$$

oribus pro M et N restituendis

$$pz = \frac{nE^nF^n(F^2 - E^2)(EEFF + 1)}{4EEFF(F^{2n} - E^{2n})} - \frac{F^{2n} + E^{2n}}{4E^nF^n}$$

$$pp + zz = \frac{n(F^2 - E^2)(EEFF + 1)(F^{2n} + E^{2n})}{4EEFF(F^{2n} - E^{2n})} - 1.$$

COROLLARIUM 2

t $n = 1$, erit

$$= \sqrt{\left(\frac{EEFF + 1}{4EEFF} + \frac{1}{2EF}\right)} = \frac{EF + 1}{2EF} \quad \text{et} \quad N = \frac{EF - 1}{2EF},$$

$$F + \frac{1}{2} E - \frac{1}{2E} - \frac{1}{2F} \quad \text{et} \quad z - p = \frac{1}{2} F - \frac{1}{2} E + \frac{1}{2E} - \frac{1}{2F}$$

$$= E - \frac{1}{E} \quad \text{seu} \quad p = e \quad \text{et} \quad 2z = F - \frac{1}{F} \quad \text{seu} \quad z = f,$$

t p et z in puncta e et f incidunt.

COROLLARIUM 3

83. Si arcus pz debeat esse duplus arcus dati ef

$$M = \sqrt{\left(\frac{EEFF+1}{2EEFF(FF+EE)} + \frac{1}{2EEFF}\right)} = \sqrt{\frac{(EF}{2EE}}$$

et

$$N = \sqrt{\left(\frac{EEFF+1}{2EEFF(FF+EE)} - \frac{1}{2EEFF}\right)} = \sqrt{\frac{(EF}{2EE}}$$

unde, si arcus ef in vertice A terminetur, ut sit
$M = \frac{1}{F}$ et $N = 0$; sicque prodit $z + p = 0$ et $z - p =$
$p = -f$ et $z = +f$. Hoc ergo casu arcus pz medium
et utrinque arcum ipsi ef seu Af aequalem complectit

COROLLARIUM 4

84. Si arcus pz debeat esse triplus arcus ef seu

$$M = \sqrt{\left(\frac{3(EEFF+1)}{4EEFF(F^4+E^2F^2+E^4)} + \frac{}{2E}\right.}$$

sive

$$M = \sqrt{\frac{3E^3F^3+3EF+2F^4+2EEFF+}{4E^3F^3(F^4+EEFF+E^4)}}$$

et

$$N = \sqrt{\frac{3E^3F^3+3EF-2F^4-2EEFF-}{4E^3F^3(F^4+EEFF+E^4)}}$$

COROLLARIUM 5

85. Si hoc casu, quo $n = 3$, arcus ef in vertice
et $E = 1$, unde

$$M = \sqrt{\frac{2F^4+3F^3+2FF+3F+2}{4F^3(F^4+F^2+1)}}$$

sive

$$M = (F+1)\sqrt{\frac{2FF-F+2}{4F^3(F^4+F^2+1)}}$$

et

$$N = (F-1)\sqrt{\frac{-2FF-F-2}{4F^3(F^4+F^2+1)}},$$

qui ergo valor est imaginarius.

COROLLARIUM 6

rgo arcus ef triplum exhiberi possit, is non in vertice A termi-
n E debet esse maius quam 1 atque adeo limes dabitur, infra
nequeat. Ad quem limitem inveniendum resolvi oportet hanc

$$3E^3F^3 + 3EF = 2F^4 + 2EEFF + 2E^4.$$

ponatur $EF = S$ et $EE + FF = R$; erit

$$3S = 2RR - 2SS \quad \text{ideoque} \quad R = \sqrt{\left(\tfrac{3}{2}S^3 + SS + \tfrac{3}{2}S\right)},$$

$$F + E = \sqrt{\left(2S + \sqrt{\left(\tfrac{3}{2}S^3 + SS + \tfrac{3}{2}S\right)}\right)},$$

$$F - E = \sqrt{\left(-2S + \sqrt{\left(\tfrac{3}{2}S^3 + SS + \tfrac{3}{2}S\right)}\right)}.$$

$E > 1$ et $F > 1$, debet esse $R > 2$ et $3S^3 + 2SS + 3S > 8$
.

COROLLARIUM 7

ratim ergo pro casu $n = 3$ oportet sit

$$S > 2RR - 2SS \quad \text{ideoque} \quad R < \sqrt{\left(\tfrac{3}{2}S^3 + SS + \tfrac{3}{2}S\right)};$$

numerus unitate minor, reperitur

$$F + E = \sqrt{\left(2S + \alpha\sqrt{\left(\tfrac{3}{2}S^3 + SS + \tfrac{3}{2}S\right)}\right)},$$

$$F - E = \sqrt{\left(-2S + \alpha\sqrt{\left(\tfrac{3}{2}S^3 + SS + \tfrac{3}{2}S\right)}\right)}.$$

so $\alpha\alpha > \frac{8S}{3SS + 2S + 3}$ et $S > 1$.

COROLLARIUM 8

mus $S = 2$; erit $\alpha\alpha > \frac{16}{19}$. Capiatur $\alpha = 1$, ut sit $EF = 2$ et
$\sqrt{19}$; erit

$$E = \sqrt{(\sqrt{19} + 4)}, \quad E = \tfrac{1}{2}\sqrt{(\sqrt{19} + 4)} - \tfrac{1}{2}\sqrt{(\sqrt{19} - 4)},$$

$$E = \sqrt{(\sqrt{19} - 4)}, \quad F = \tfrac{1}{2}\sqrt{(\sqrt{19} + 4)} + \tfrac{1}{2}\sqrt{(\sqrt{19} - 4)};$$

ergo

$$e = \tfrac{1}{8}\sqrt{(\sqrt{19}+4)} - \tfrac{3}{8}\sqrt{(\sqrt{19}-4)}$$

et

$$f = \tfrac{1}{8}\sqrt{(\sqrt{19}+4)} + \tfrac{3}{8}\sqrt{(\sqrt{19}-4)}$$

Porro reperitur

$$M = \frac{1}{2\sqrt{2}} \quad \text{et} \quad N = 0;$$

unde

$$z = -p = \frac{1}{4\sqrt{2}}(2+\sqrt{19})\sqrt{(\sqrt{19}-4}$$

hic ergo arcus triplus utrinque circa verticem aequalit

III. DE COMPARATIONE SUPERFICIERUM SPHAE COMPRESSI ET CONOIDIS HYPERB

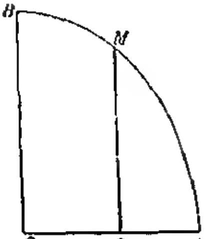


Fig. 4.

89. Sit igitur primum proposit cum genitum rotatione ellipsis BI minorem AC. Ponatur semiaxis m axis maior $CB = a\sqrt{m}$ existente m Sumta iam in axe minore a cent erit applicata $PM = \sqrt{m(aa - xx)}$, cum $= dx\sqrt{\frac{aa + (m-1)xx}{aa - xx}}$.

90. Posita nunc ratione diametri ad peripheriam = ficiei sphaeroidicae a revolutione arcus BM genita, s scissae $CP = x$, aequalis huic integrali $2\pi\int dx\sqrt{m(aa}$ hoc integrale, quod tanquam functio abscissae x spect

$$\int dx\sqrt{m(aa+(m-1)xx)} = \Pi . x$$

91. Portio ergo superficiei sphaeroidicae ellipt respondens erit $= 2\pi \cdot \Pi . x$, ubi functio $\Pi . x$, uti perspi seu rectificatione parabolae pendet, eritque $\Pi . x = 0$ ponatur $x = a$, tum $2\pi \cdot \Pi . a$ exhibebit semissem toti

92. Sit porro conoides hyperbolicum genitum re (Fig. 5, p. 145) circa suum axem cap, cuius centrum

versus $ca = c$, semiaxis autem $\sqrt{n}$. Sumta ergo in axe a centro anque $cp = y$, quae quidem sit icata $pm = \sqrt{n(yy-cc)}$ et ele- olicum $= dy\sqrt{\frac{(n+1)yy-cc}{yy-cc}}$.

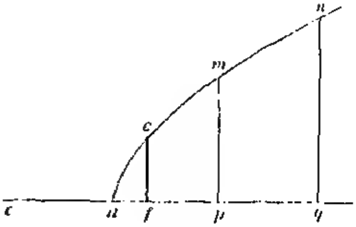


Fig. 5.

erit portio superficiei conoidis istius hyperbolici ex arcu am issae $cp = y$ respondens $= 2\pi\int dy\sqrt{n((n+1)yy-cc)}$. Quod pectari possit tanquam functio ipsius y, ita indicetur

$$\int dy\sqrt{n((n+1)yy-cc)} = \Theta.y$$

si capiatur $y = c$. Erit ergo superficies conoidis hyperbolici y respondens $= 2\pi\cdot\Theta.y$.

rentur hae binae formulae cum illis, quae supra § 38 sunt um sit

$$\Pi.x = \int\frac{dx(aa+(m-1)xx)\sqrt{m}}{\sqrt{(aa+(m-1)xx)}},$$

$$A = aa,\quad C = m-1,$$

$-1) = aa\sqrt{m}$ et $\frac{m-1}{aa}\mathfrak{B}\sqrt{(m-1)} = (m-1)\sqrt{m}$;

$$\mathfrak{A} = \frac{aa\sqrt{m}}{\sqrt{(m-1)}} \quad\text{et}\quad \mathfrak{B} = \frac{aa\sqrt{m}}{\sqrt{(m-1)}}.$$

pro hyperbola cum sit

$$\Theta.y = \int\frac{dy(-cc+(n+1)yy)\sqrt{n}}{\sqrt{(-cc+(n+1)yy)}},$$

t $F = n+1$ eritque ob $\mathfrak{C} = 0$

$$y\left(\mathfrak{A}+\frac{F}{E}\mathfrak{B}yy\right)\sqrt{F} \over \sqrt{(E+Fyy)}} = \frac{aa\sqrt{m(n+1)}}{\sqrt{(m-1)}}\int\frac{dy\left(-1+\frac{(n+1)yy}{cc}\right)}{\sqrt{(-cc+(n+1)yy)}},$$

ergo

$$-\int \frac{dy\left(\mathfrak{A} + \frac{F}{E}\mathfrak{B}yy\right)\sqrt{F}}{\sqrt{(E+Fyy)}} = \frac{aa\sqrt{m(n+}}{cc\sqrt{n(m-}}$$

96. His ergo substitutionibus factis habebimus

$$\Pi . x + \frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}}\Theta . y = \text{Const.} + \frac{(n+1)}{}$$

cui satisfacit haec relatio inter x et y

$$\frac{aay}{m-1} = k\sqrt{\left(\frac{aa}{m-1} + xx\right)} - x\sqrt{\left(kk - \right.}$$

seu

$$\frac{ccx}{n+1} = k\sqrt{\left(-\frac{cc}{n+1} + yy\right)} + y\sqrt{\left(kk - \right.}$$

ubi $\sqrt{\left(kk - \frac{aacc}{(m-1)(n+1)}\right)}$ negative accipi conveniet.

97. Vel ponatur $k = \frac{ae}{\sqrt{(m-1)}}$, et si fuerit

$$y = \frac{e}{a}\sqrt{(aa + (m-1)xx)} + \frac{x\sqrt{(m-1)}}{a\sqrt{(n+1)}}\sqrt{((}$$

seu

$$x = \frac{ae\sqrt{(n+1)}}{ce\sqrt{(m-1)}}\sqrt{((n+1)yy - cc)} - \frac{ay\sqrt{(n+1)}}{cc\sqrt{(m-1)}}$$

erit

$$\Pi . x + \frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}}\Theta . y = \text{Const.} + \frac{(n+}{}$$

98. Ad constantem autem definiendam ponatu
eritque $y = e$, unde prodit

$$\text{Const.} = \frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}}\Theta . e;$$

sicque habebitur

$$\Pi . x + \frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}}(\Theta . y - \Theta . e) = \frac{(n+}{}$$

At si in hyperbola capiatur abscissa $cf = e$, erit s
em nata $= 2\pi \cdot (\Theta . y - \Theta . e)$.

oniam igitur y per x determinatur, erit quoque

$$\cdots cc) = \frac{c}{a} x \sqrt{(m-1)(n+1)} + \frac{1}{a} \sqrt{(aa+(m-1)xx)((n+1)ee-cc)},$$

$$(n+1)yy-cc) = \left(\frac{c}{a} + \frac{\delta}{a}\sqrt{((n+1)ee-cc)}\right)\sqrt{(aa+(m-1)xx)}$$

$$+ x\left(\frac{\delta c}{a}\sqrt{(m-1)(n+1)} + \frac{\sqrt{(m-1)}}{a\sqrt{(n+1)}}\sqrt{((n+1)ee-cc)}\right);$$

$$1 : \delta\sqrt{(m-1)(n+1)} = \delta : \frac{\sqrt{(m-1)}}{\sqrt{(n+1)}};$$

$$\delta = \frac{1}{\sqrt{(n+1)}}$$

inetur

$$\sqrt{((n+1)yy-cc)} + y\sqrt{(n+1)}$$

$$\frac{\cdot+1)}{a} + \frac{1}{a}\sqrt{((n+1)ee-cc)}\right)\left(\sqrt{(aa+(m-1)xx)} + x\sqrt{(m-1)}\right).$$

Datis ergo abscissis $CP = x$ et $cf = c$ abscissa $cp = y$ ita definiri t

$$\frac{(n+1)yy-cc) + y\sqrt{(n+1)}}{(n+1)ee-cc) + c\sqrt{(n+1)}} = \sqrt{\left(1 + \frac{(m-1)xx}{aa}\right)} + \frac{x}{a}\sqrt{(m-1)}.$$

em est

$$\frac{ay\sqrt{((n+1)yy-cc)}}{2\sqrt{(m-1)(n+1)}} = \frac{ac\sqrt{(n+1)ee-cc)}}{2\sqrt{(m-1)(n+1)}} + \frac{ccx\sqrt{(aa+(m-1)xx)}}{2a(n+1)}.$$

PROBLEMA HUGENIANUM

Dato sphaeroide elliptico lato ABC invenire conoides hyperbolicum apm, llus describi possit geometrice, cuius area aequalis sit futura utrique phaeroidicae et conoidicae iunctim sumtae.[1])

[1]) notam 1 p. 111. A. K.

Manentibus pro utroque corpore denominationibu
tuatur

$$\frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}} = 1 \quad \text{seu} \quad cc = \frac{aa\sqrt{m(n}}{\sqrt{n(m-}}$$

unde semiaxis transversus hyperbolae c determinatu
specie arbitrio nostro relicta, eritque stabilita superior

$$H.x+(\Theta.y-\Theta.e) = \frac{(n+1)ac\sqrt{m}}{cc}xy = \frac{cxy\sqrt{n}}{}$$

102. Cum nunc sit superficies sphaeroidis ex
Sup. $BM = 2\pi \cdot H.x$ et superficies conoidis ex
Sup. $em = 2\pi(\Theta.y-\Theta.e)$, erit

$$\text{Sup. } BM + \text{Sup. } em = \frac{2\pi exy\sqrt{n(m-1)(}}{a}$$

Unde si hae duae superficies iunctim sumtae aequentu
$=r$, ob eius aream $=\pi rr$ erit

$$rr = \frac{2exy\sqrt{n(m-1)(n+1)}}{a}.$$

103. Hic iam continetur solutio problematis sensu
Casu enim Hugeniano, quo integrum sphaeroides assun
redit, eius semissis, erit $x=a$; tum vero punctum e in
unde fit $e=c$. Erit ergo hoc casu

$$y = c\sqrt{m} + \frac{c\sqrt{n(m-1)}}{\sqrt{(n+1)}} = cp$$

fietque

$$\text{Sup. } BA + \text{Sup. } am = 2\pi(n+1)aa \cdot \frac{m+\sqrt{n}}{\sqrt{(}}$$

104. Radio ergo circuli utrique superficiei simul a

$$rr = 2aa(m(n+1)+\sqrt{mn(m-1)(n-}$$

sive

$$r = a\sqrt{2(\sqrt{m(n+1)}+\sqrt{n(m-1)})\sqrt{m(n}}$$

$$cp = y = \frac{c}{\sqrt{(n+1)}}\left(\sqrt{m(n+1)} + \sqrt{n(m-1)}\right);$$

i debet

$$c = a\sqrt{\frac{m(n+1)}{n(m-1)}}.$$

o simplicissima Problematis HUGENIANI.

SOLUTIO SECUNDA

relatio inter x et y sit ita comparata, ut sit

$$\frac{1)yy - cc) + y\sqrt{(n+1)}}{1)cc - cc) + c\sqrt{(n+1)}} = \sqrt{\left(1 + \frac{(m-1)xx}{aa}\right)} + \frac{x}{a}\sqrt{(m-1)}$$

$$\Pi . x + \frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}}(\Theta . y - \Theta . c)$$

$$\frac{y\sqrt{((n+1)yy - cc)}}{\sqrt{(m-1)}} - \frac{aac\sqrt{((n+1)cc - cc)}}{\sqrt{(m-1)}} + \frac{ccx\sqrt{(aa + (m-1)xx)}}{\sqrt{(n+1)}}\Big),$$

noide nova abscissa $cq = z$ et pro c iam sumatur y, ut sit

$$\frac{1)zz - cc) + z\sqrt{(n+1)}}{1)yy - cc) + y\sqrt{(n+1)}} = \sqrt{\left(1 + \frac{(m-1)xx}{aa}\right)} + \frac{x}{a}\sqrt{(m-1)};$$

$$\Pi . x + \frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}}(\Theta . z - \Theta . y)$$

$$\frac{az\sqrt{((n+1)zz - cc)}}{\sqrt{(m-1)}} - \frac{aay\sqrt{((n+1)yy - cc)}}{\sqrt{(m-1)}} + \frac{ccx\sqrt{(aa + (m-1)xx)}}{\sqrt{(n+1)}}\Big).$$

ntur hae formulae invicem atque y prorsus eliminabitur; fiet

$$\frac{)zz - cc) + z\sqrt{(n+1)}}{1)cc - cc) + c\sqrt{(n+1)}} = \left(\sqrt{\left(1 + \frac{(m-1)xx}{aa}\right)} + \frac{x}{a}\sqrt{(m-1)}\right)^2$$

$$2\Pi . x + \frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}}(\Theta . z - \Theta . c)$$

$$\frac{az\sqrt{((n+1)zz - cc)}}{\sqrt{(m-1)}} - \frac{aac\sqrt{((n+1)cc - cc)}}{\sqrt{(m-1)}} + \frac{2ccx\sqrt{(aa + (m-1)xx)}}{\sqrt{(n+1)}}\Big).$$

107. Statuatur iam

$$\frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}}=2 \quad \text{seu} \quad cc=\frac{aa\sqrt{m(}{2\sqrt{n(m}$$

erit per $\frac{2\pi}{2}$ multiplicando

$$\text{Sup. } BM+\text{Sup. } en$$

$$=\frac{\pi\sqrt{m(n+1)}}{2cc}\left(\frac{aaz\sqrt{((n+1)zz-cc)}}{\sqrt{(m-1)}}-\frac{aac\sqrt{((n+1)cc-cc)}}{\sqrt{(m-1)}}\right.$$

unde facile radius circuli aequalis definitur.

108. Sit nunc pro casu HUGENIANO $x=a$ et $c=$

$$\frac{\sqrt{((n+1)zz-cc)}+z\sqrt{(n+1)}}{c\left(\sqrt{n}+\sqrt{(n+1)}\right)}=\left(\sqrt{m}+\sqrt{(}\right.$$

Hincque invento z existenteque

$$cc=\frac{aa\sqrt{m(n+1)}}{2\sqrt{n(m-1)}}$$

erit

$$\text{Sup. } BA+\text{Sup. } an=\frac{\pi\sqrt{m(n+1)}}{2cc}\left(\frac{aaz\sqrt{((n+1)zz-cc)}}{\sqrt{(m-1)}}\right.$$

SOLUTIO GENERALIS

109. Si hac ratione continuo ulterius progrediamu est factum, reperietur, si abscissa $cq=z$ existente $c/$

$$\frac{\sqrt{((n+1)zz-cc)}+z\sqrt{(n+1)}}{\sqrt{((n+1)cc-cc)}+c\sqrt{(n+1)}}=\left(\sqrt{\left(1+\frac{(m-1)xx}{aa}\right.}\right.$$

fore

$$\mu\Pi.x+\frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}}(\Theta.z-\Theta.$$

$$=\frac{\sqrt{m(n+1)}}{2cc}\left(\frac{aaz\sqrt{((n+1)zz-cc)}}{\sqrt{(m-1)}}-\frac{aac\sqrt{((n+1)cc-cc)}}{\sqrt{(m-1)}}\right.$$

$$=\frac{\mu}{2\pi}\text{ Sup. } BM+\frac{aa\sqrt{m(n+1)}}{2\pi cc\sqrt{n(m-1)}}\text{ S}$$

Pro casu ergo HUGENII posito $x = a$ et $e = c$ fiat $\frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}} = \mu$ et abscissa $cq = z$, ita ut sit

$$\frac{\sqrt{((n+1)zz-cc)}+z\sqrt{(n+1)}}{c(\sqrt{n}+\sqrt{(n+1)})} = (\sqrt{m}+\sqrt{(m-1)})^{\mu},$$

$$+ \text{Sup. } an = \frac{\pi\sqrt{m(n+1)}}{\mu cc}\left(\frac{aaz\sqrt{((n+1)zz-cc)}}{\sqrt{(m-1)}} - \frac{aacc\sqrt{n}}{\sqrt{(m-1)}} + \frac{\mu aacc\sqrt{m}}{\sqrt{(n+1)}}\right)$$

$$BA + \text{Sup. } an = \pi\left(z\sqrt{n((n+1)zz-cc)} - ncc + \frac{\mu cc\sqrt{mn(m-1)}}{\sqrt{(n+1)}}\right)$$

$$= \pi\left(z\sqrt{n((n+1)zz-cc)} - ncc + maa\right).$$

Quaecumque ergo fuerit hyperbola, ex qua conoides nascitur, dummodo $\frac{\ldots n+1)}{\ldots 1)} = \mu$ numerus rationalis, ab eo semper portio an abscindi cuius superficies ad superficiem sphaeroidis BMA addita per circhiberi potest, cuius radius r geometrice est assignabilis; erit enim

$$r = \sqrt{\left(maa - ncc + z\sqrt{n((n+1)zz-cc)}\right)}.$$

Quo autem facilius pateat, quomodo abscissa $cq = z$ reperiri debeat,

$$\ldots\left(\frac{n+1)zz}{cc} - 1\right) + \frac{z}{c}\sqrt{(n+1)} = \left(\sqrt{(n+1)}+\sqrt{n}\right)\left(\sqrt{m}+\sqrt{(m-1)}\right)^{\mu},$$

$$\ldots(n+1) - \sqrt{\left(\frac{(n+1)zz}{cc} - 1\right)} = \left(\sqrt{(n+1)}-\sqrt{n}\right)\left(\sqrt{m}-\sqrt{(m-1)}\right)^{\mu};$$

...de tam z quam $\sqrt{((n+1)zz-cc)}$ colligentur.

Hinc autem porro concluditur fore

$$\ldots((n+1)zz-cc) = \frac{cc\sqrt{n}}{4\sqrt{(n+1)}}\left(\sqrt{(n+1)}+\sqrt{n}\right)^2\left(\sqrt{m}+\sqrt{(m-1)}\right)^{2\mu}$$

$$- \frac{cc\sqrt{n}}{4\sqrt{(n+1)}}\left(\sqrt{(n+1)}-\sqrt{n}\right)^2\left(\sqrt{m}-\sqrt{(m-1)}\right)^{2\mu}.$$

Ac si ponatur brevitatis gratia

$$\sqrt{m} + \sqrt{(m-1)} = M \quad \text{et} \quad \sqrt{n} + \sqrt{(n+1)} =$$

erit

$$z = \frac{c}{2\sqrt{(n+1)}}(M^{\nu} N + M^{-\nu} N^{-1})$$

et

$$r = \sqrt{\Big(maa + \frac{cc\sqrt{n}}{4\sqrt{(n+1)}}(M^{\nu} - M^{-\nu})(M^{\nu} N^2 + M^{-}$$

sicque problema non difficulter construitur, dummodo e
rationalis.

114. Haec igitur exempla sufficiant usum novae methodi,
ostendisse; etsi enim haec eadem exempla methodo consuet
tamen non solum ad calculos admodum intricatos deveniri
integratione, qua formulae differentiales vel ad quadratura
logarithmos reducantur, absolute est opus. Huius igitur r
signe commodum in hoc consistit, quod eius beneficio eado
sine laborioso calculo quam sine ulla integratione resolvi
causam inde merito multo maiora ac subliniora expectar
omnium consuetarum methodorum penitus superent.

SPECIMEN ALTERUM METHODI NOVAE QUANTITATES TRANSCENDENTES INTER SE COMPARANDI DE COMPARATIONE ARCUUM ELLIPSIS

Commentatio 261 indicis Enestroemiani
Novi commentarii academiae scientiarum Petropolitanae 7 (1758/9), 1761, p. 3—48
Summarium (Commentationum 261 et 263) ibidem p. 5—8[1])

1. Primum huius methodi specimen, quod nuper[2]) exhibui, in compara
uum circuli et parabolae conicae versabatur; quae comparatio etsi i
ectata non est nova, cum methodis vulgaribus iam pridem sit expe
non inde exordiendum est visum, quo novae huius methodi, quam
brui, vis melius perspiciatur; quod non solum ad easdem veritates,
thodis consuetis erui solent, perducat, sed etiam viam longe faciliore
editiorem eadem praestandi patefaciat. Methodus enim consueta
tiones satis taediosas requirit atque ita est comparata, ut, nisi a
arum curvarum, qui inter se sunt comparandi, ad quadraturas cog
culi ac hyperbolae revocari potuissent, nullo modo in subsidium ve
nisset.

2. Quantum ergo haec nova methodus praestare valeat, uberius ex
ratione arcuum ellipsis et hyperbolae perspicietur; quarum curvarum re
io cum nullo modo neque ad circuli quadraturam neque ad logarit
uci queat, methodis consuetis nullus amplius locus relinquitur neque
modus patet diversos istarum curvarum arcus inter se conferendi. Q

1) Vide p. 108. A. K.

2) L. Euleri Commentatio 263 (indicis Enestroemiani); vide p. 108. A. K.

corum et hyperbolicorum pari cum successu institu
parabolicorum, quoniam methodi vulgares ad id plane
summus usus novae methodi inde elucebit.

3. Inveni autem huius methodi ope arcus tam ellip
pari modo inter se comparari posse atque arcus para
mento esse, quod harum curvarum rectificatio vires A
gredi videatur. Quin etiam haec comparatio sub iisd
in parabola institui potest, ita ut proposito sive in e
arcu quocunque ab alio quovis eiusdem curvae punct
qui ab illo differat quantitate geometrice assignabili.
puncto quovis arcus exhiberi poterit, qui ab arcu p
vel toties sumto, quoties lubuerit, quantitate geometr

4. Porro autem effici potest, ut haec differentia
arcusque inventus ipsi arcui proposito eiusve multi
perinde atque in parabola id fieri posse notum est.
venit, ut bini arcus aequales exhiberi nequeant, qui
similes; verum hoc multo magis notatu erit dignu
quam hyperbola proposito arcu quocunque semper al
qui illius duplo vel triplo vel multiplo cuicunque sit

5. Quemadmodum igitur ratione comparationis di
et hyperbola indolem parabolae sequuntur, ita curva
similis deprehenditur. In ea enim curva aeque ac
fuerit arcus quicunque, a puncto quovis dato arcu
proposito vel fuerit aequalis vel duplo maior vel t
lubuerit. In hac namque curva perinde atque in circ
dantur, quorum differentia geometrice possit assignar

6. Quae autem hic sum allaturus, multo latius
commemoratas, ellipsin, hyperbolam et lemniscatan
casus quasi simplicissimos constituunt formularum, q
peditat. His enim formulis evolutis similem compar
curvarum generibus instituere licebit. Quemadmodum

huius aequationis innitebatur

$$0 = \alpha + 2\beta(x + y) + \gamma(xx + yy) + 2\delta xy,$$

uationem latius patentem fundamenti loco assumi oportet, ex qua
que variabilis ope extractionis radicis quadratae definiri queat. Si
osita haec

AEQUATIO CANONICA

$$0 = \alpha + \gamma(xx + yy) + 2\delta xy + \zeta xxyy$$

odsi ex hac aequatione tam valorem ipsius x quam ipsius y seorsim
, obtinebimus

$$y = \frac{-\delta x + \sqrt{(\delta\delta xx - (\alpha + \gamma xx)(\gamma + \zeta xx))}}{\gamma + \zeta xx},$$

$$x = \frac{-\delta y - \sqrt{(\delta\delta yy - (\alpha + \gamma yy)(\gamma + \zeta yy))}}{\gamma + \zeta yy},$$

radicalibus diversa tribuimus signa, quoniam ab arbitrio nostro
ummodo eorum in sequentibus debita ratio teneatur.

namus, ut brevitati consulamus, has formulas surdas

$$\sqrt{(\delta\delta xx - (\alpha + \gamma xx)(\gamma + \zeta xx))} = X$$

$$\sqrt{(\delta\delta yy - (\alpha + \gamma yy)(\gamma + \zeta yy))} = Y,$$

us

$$y = \frac{-\delta x + X}{\gamma + \zeta xx} \quad \text{seu} \quad X = \gamma y + \delta x + \zeta xxy,$$

$$x = \frac{-\delta y - Y}{\gamma + \zeta yy} \quad \text{seu} \quad -Y = \gamma x + \delta y + \zeta xyy.$$

nc aequatio canonica etiam differentietur eritque

$$0 = dx(\gamma x + \delta y + \zeta xyy) + dy(\gamma y + \delta x + \zeta xxy),$$

gimus fore

$$0 = -Ydx + Xdy \quad \text{sive} \quad \frac{dy}{Y} - \frac{dx}{X} = 0.$$

X sit functio ipsius x et Y ipsius y, erit integrando

$$\int \frac{dy}{Y} - \int \frac{dx}{X} = \text{Const.}$$

10. Vicissim ergo novimus, si huiusmodi aequatio inte
posita

$$\int \frac{dy}{Y} - \int \frac{dx}{X} = \text{Const.},$$

in qua X et Y eiusmodi functiones irrationales ipsarum
ut sit

$$X = \sqrt{(\delta\delta xx - (\alpha + \gamma xx)(\gamma + \zeta xx))}$$

et

$$Y = \sqrt{(\delta\delta yy - (\alpha + \gamma yy)(\gamma + \zeta yy))},$$

tum huic aequationi satisfacere relationem inter x et y
canonicam definitam.

11. Quemadmodum autem invenimus aequationem $\frac{dy}{Y}$ —
sideremus nunc aequationem latius patentem

$$\frac{Q\,dy}{Y} - \frac{P\,dx}{X} = dV$$

et investigemus, cuiusmodi functiones P et Q esse queant
ut dV integrationem admittat ideoque differentia formularu

$$\int \frac{Q\,dy}{Y} - \int \frac{P\,dx}{X} = \text{Const.} + V$$

algebraice exhiberi queat.

13.[1]) Quo haec investigatio facilius institui queat, ponar
$x\,dy + y\,dx = du$ habebimus $dy = \frac{du}{x} - \frac{y\,dx}{x}$, qui valor loco
differentiali substitutus dabit

$$0 = dx(\gamma x + \delta y + \zeta xyy) + \frac{du}{x}(\gamma y + \delta x + \zeta xxy) - dx\Big(\frac{\gamma y}{x}$$

seu per x multiplicando

$$0 = dx(\gamma xx - \gamma yy) + du(\gamma y + \delta x + \zeta xxy)$$

seu

$$0 = \gamma\,dx(xx - yy) + X\,du.$$

1) In editione principe loco numerum 12 et qui sequuntur falso nume
scripti sunt. Falsos paragraphorum numeros retinendos esse putavimus.

ergo $\frac{dx}{X} = \frac{du}{\gamma(yy - xx)}$, et cum sit $\frac{dy}{Y} = \frac{dx}{X}$, erit quoque $\frac{dy}{Y} = \frac{du}{\gamma(yy - xx)}$,
...ius

$$dV = \frac{(Q - P)du}{\gamma(yy - xx)}.$$

...patet, si sit $Q = yy$ et $P = xx$, fore

$$dV = \frac{du}{\gamma} \quad \text{et} \quad V = \frac{u}{\gamma} = \frac{xy}{\gamma}.$$

...aequatione canonica erit

$$\int \frac{yy\,dy}{Y} - \int \frac{xx\,dx}{X} = \text{Const.} + \frac{xy}{\gamma}.$$

...lis autem integratio quantitatis V quoque succedit, si pro P et Q
...potestates quaevis parium dimensionum ipsarum x et y. Quod
... ponamus $xx + yy = t$ et ob $xy = u$ aequatio canonica abit in

$$0 = \alpha + \gamma t + 2\delta u + \zeta uu,$$

$\ldots \frac{-\alpha - 2\delta u - \zeta uu}{\gamma}.$

...amus iam $P = x^4$ et $Q = y^4$; erit

$\ldots \frac{du}{\gamma}(xx + yy) = \frac{t\,du}{\gamma} \quad \text{ideoque} \quad dV = \frac{-du}{\gamma\gamma}(\alpha + 2\delta u + \zeta uu);$

...ndo fit

$\ldots \frac{-\alpha u}{\gamma\gamma} - \frac{\delta uu}{\gamma\gamma} - \frac{\zeta u^3}{3\gamma\gamma} \quad \text{sive} \quad V = \frac{-xy}{3\gamma\gamma}(3\alpha + 3\delta xy + \zeta xxyy).$

$\ldots y = -\alpha - \gamma(xx + yy) - 2\delta xy$ habebitur

$$V = \frac{-xy}{3\gamma\gamma}(2\alpha - \gamma(xx + yy) + \delta xy).$$

...re nostra aequatio canonica etiam satisfaciet huic aequationi

$$\int \frac{y^4\,dy}{Y} - \int \frac{x^4\,dx}{X} = \text{Const.} - \frac{xy}{3\gamma\gamma}(3\alpha + 3\delta xy + \zeta xxyy).$$

Atque his tribus casibus coniunctim aequatio
tioni differentiali latius patenti

$$\int \frac{dy(\mathfrak{A} + \mathfrak{B}yy + \mathfrak{C}y^4)}{\sqrt{(\delta\delta yy - (\alpha + \gamma yy)(\gamma + \zeta yy))}} - \int \frac{dx(\mathfrak{A} + \mathfrak{B}}{\sqrt{(\delta\delta xx - (\alpha +}}$$

$$= \text{Const.} + \frac{\mathfrak{B}xy}{\gamma} - \frac{\mathfrak{C}xy}{3\gamma\gamma}(3\alpha + 3\delta xy +$$

18. Si ulterius progredi velimus, ponamus $P = x$

$$dV = \frac{du}{\gamma}(y^4 + xxyy + x^4) = \frac{du}{\gamma}(tt -$$

substituto ergo pro t valore invento erit

$$dV = \frac{du}{\gamma^3}\Big(\alpha\alpha + 4\alpha\delta u + (4\delta\delta + 2\alpha\zeta - \gamma\gamma)uu -$$

ideoque integrando

$$V = \frac{u}{\gamma^3}\Big(\alpha\alpha + 2\alpha\delta u + \frac{1}{3}(4\delta\delta + 2\alpha\zeta - \gamma\gamma)uu +$$

Unde erit per aequationem canonicam

$$\int \frac{y^6 dy}{Y} - \int \frac{x^6 dx}{X}$$

$$= \text{Const.} + \frac{xy}{15\gamma^3}\Big(15\alpha\alpha + 30\alpha\delta xy + 5(4\delta\delta + 2\alpha\zeta - \gamma\gamma)xx$$

19. Nunc autem formulis nostris irrationalibus X
inducamus, quae facilius ad quosvis casus accommoda

$$X = \sqrt{p}(A + Cxx + Ex^4) \quad \text{et} \quad Y = \sqrt{p}(A +$$

necesse ergo est sit

$$Ap = -\alpha\gamma, \quad Ep = -\gamma\zeta, \quad Cp = \delta\delta -$$

unde fit

$$\alpha = \frac{-Ap}{\gamma}, \quad \zeta = \frac{-Ep}{\gamma} \quad \text{et} \quad \delta = \sqrt{\Big(\gamma\gamma + C}$$

a $\gamma\gamma = A$ et $p = kk$ sumaturque $\gamma = -\sqrt{A}$ ac fiet

A, $\gamma = -\sqrt{A}$, $\zeta = \frac{Ekk}{\sqrt{A}}$ et $\delta = \sqrt{(A + Ckk + Ek^4)}$

$k\sqrt{(A + Cxx + Ex^4)}$ et $Y = k\sqrt{(A + Cyy + Ey^4)}$

nica prodibit

$$-A(xx + yy) + 2xy\sqrt{A(A + Ckk + Ek^4)} + Ekkxxyy.$$

autem variabiles x et y ita a se invicem pendent, ut sit

$$= -y\sqrt{A} + x\sqrt{(A + Ckk + Ek^4)} + \frac{Ekk}{\sqrt{A}}xxy,$$

$$= x\sqrt{A} - y\sqrt{(A + Ckk + Ek^4)} - \frac{Ekk}{\sqrt{A}}xyy,$$

$$y = \frac{x\sqrt{A(A + Ckk + Ek^4)} - k\sqrt{A(A + Cxx + Ex^4)}}{A - Ekkxx},$$

$$x = \frac{y\sqrt{A(A + Ckk + Ek^4)} + k\sqrt{A(A + Cyy + Ey^4)}}{A - Ekkyy}.$$

tur valores satisfacient huic aequationi integrali latissime deductae, dum ea per $-k$ multiplicatur,

$$\int\frac{dx(\mathfrak{A} + \mathfrak{B}xx + \mathfrak{C}x^4)}{\sqrt{(A + Cxx + Ex^4)}} - \int\frac{dy(\mathfrak{A} + \mathfrak{B}yy + \mathfrak{C}y^4)}{\sqrt{(A + Cyy + Ey^4)}}$$

$$\frac{kxy}{\sqrt{A}} + \frac{\mathfrak{C}kxy}{3A\sqrt{A}}\left(3Akk + 3xy\sqrt{A(A + Ckk + Ek^4)} + Ekkxxyy\right)$$

$$. + \frac{\mathfrak{B}kxy}{\sqrt{A}} + \frac{\mathfrak{C}kxy}{6A\sqrt{A}}\left(3Akk + 3A(xx + yy) - Ekkxxyy\right).$$

ergo curva quaepiam ita fuerit comparata, ut abscissae x s

$$= \int\frac{dx(\mathfrak{A} + \mathfrak{B}xx + \mathfrak{C}x^4)}{\sqrt{(A + Cxx + Ex^4)}}$$

isque notetur per $\Pi.x$ et arcus alii abscissae y respo

$$\int\frac{dy(\mathfrak{A}+\mathfrak{B}yy+\mathfrak{C}y^4)}{\sqrt{(A+Cyy+Ey^4)}}$$

per $\Pi.y$, inter hos duos arcus ista relatio locum habe

$$\Pi.x-\Pi.y=\text{Const.}+\frac{\mathfrak{B}kxy}{\sqrt{A}}+\frac{\mathfrak{C}kxy}{6A\sqrt{A}}\big(3Akk+3A(x$$

siquidem abscissae x et y ita a se invicem pendeant,

$$x=\frac{y\sqrt{A(A+Ckk+Ek^4)}+k\sqrt{A(A+Cyy}}{A-Ekkyy}$$

et

$$y=\frac{x\sqrt{A(A+Ckk+Ek^4)}-k\sqrt{A(A+Cxx}}{A-Ekkxx}$$

24. Ad istam autem constantem, quam aequatio int minandam consideretur casus, quo $y=0$ et quo fit x abscissae evanescenti conveniens quoque evanescat, $\Pi.k=\text{Const.}$, quo valore substituto habebitur

$$\Pi.x-\Pi.y-\Pi.k=\frac{\mathfrak{B}kxy}{\sqrt{A}}+\frac{\mathfrak{C}kxy(kk+xx+y}{2\sqrt{A}}$$

Hoc ergo modo terni arcus in ista curva dantur, binorum reliquorum superat quantitate geometrice assi

25. Hinc iam in genere patet, si curva ita fuerit abscissae x respondens sit

$$\Pi.x=\int\frac{\mathfrak{A}dx}{\sqrt{(A+Cxx+Ex^4)}}$$

ideoque sit $\mathfrak{B}=0$ et $\mathfrak{C}=0$, tum arcuum illorum d abire; hocque ergo casu in hac curva arcuum comp poterit atque in circulo. Sin autem in numeratore a $\mathfrak{C}x^4$ vel uterque, tum arcuum illorum ternorum differen bilis est ideoque arcuum comparatio perinde succed Ipsa autem comparatio eodem modo perficietur, quem pro circulo ac parabola exposui.

uoniam terni arcus in computum veniunt, quorum abscissae sunt , patet, quemadmodum y pendet ab x et k, eodem modo k ab ndere, unde datis binis tertia ex his aequationibus determinabitur

$$x = \frac{y\sqrt{A(A+Ckk+Ek^4)}+k\sqrt{A(A+Cyy+Ey^4)}}{A-Ekkyy},$$

$$y = \frac{x\sqrt{A(A+Ckk+Ek^4)}-k\sqrt{A(A+Cxx+Ex^4)}}{A-Ekkxx},$$

$$k = \frac{x\sqrt{A(A+Cyy+Ey^4)}-y\sqrt{A(A+Cxx+Ex^4)}}{A-Exxyy}.$$

i hinc aequatio formetur ab irrationalitate omni immunis, prodibit

$$\begin{aligned}EEk^4x^4y^4 = AA(2kkxx+2kkyy+2xxyy-k^4-x^4-y^4)\\ +4ACkkxxyy+2AEkkxxyy(kk+xx+yy).\end{aligned}$$

m ternae abscissae k, x, y pari modo sint immixtae, considerari arum quadrata kk, xx, yy tanquam radices huiusmodi aequationis

$$Z^3-pZZ+qZ-r=0,$$

$$p = kk+xx+yy,$$
$$q = kkxx+kkyy+xxyy,$$
$$r = kkxxyy,$$

$$EErr = AA(4q-pp)+4ACr+2AEpr$$

$$(Ap-Er)^2 = 4AAq+4ACr.$$

ac ergo inter coefficientes p, q et r relatione constituta si pro kk, capiantur ternae radices huius aequationis cubicae

$$Z^3-pZZ+qZ-r=0,$$

omparatione arcuum curvae, quam (§ 23) sumus contemplati,

$$\Pi.x-\Pi.y-\Pi.k = \frac{\mathfrak{B}\sqrt{r}}{\sqrt{A}}+\frac{\mathfrak{C}p\sqrt{r}}{2\sqrt{A}}-\frac{\mathfrak{C}Er\sqrt{r}}{6A\sqrt{A}}.$$

29. Sint ipsae abscissae suis signis affectae $+x$, $-y$, $-$
aequationis cubicae

$$z^3 + szz + tz - u = 0;$$

erit

$$\sqrt{r} = u, \quad q = tt + 2su \quad \text{et} \quad p = ss - 2t$$

atque

$$(Ass - 2At - Euu)^2 = 4AAtt + 8AAsu + 4AC$$

sive

$$t = \frac{Ass - Euu}{4A} - \frac{2Asu + Cuu}{Ass - Euu}.$$

Radices autem huius aequationis ope trisectionis anguli ita
sumto $v = \frac{2}{3}\sqrt{(ss - 3t)}$ et angulo Φ, cuius sit cosinus scilice

$$\cos. \Phi = \frac{27u + 9st - 2s^3}{2(ss - 3t)\sqrt{(ss - 3t)}}\text{[1]},$$

ipsae radices futurae sint

$$x = v \cos. \frac{1}{3}\Phi - \frac{1}{3}s, \qquad y = v \cos. \left(60^0 + \frac{1}{3}\Phi\right) -$$

$$k = v \cos. \left(60^0 - \frac{1}{3}\Phi\right) - \frac{1}{3}s.$$

30. Sed relictis his, quae ad radices spectant, usum for
accuratius perpendamus ac primo quidem notatu maxime dig
aequatio differentialis

$$\frac{dx}{\sqrt{(A + Cxx + Ex^4)}} = \frac{dy}{\sqrt{(A + Cyy + Ey^4)}},$$

quippe cui novimus convenire hanc aequationem integralem

$$x = \frac{y\sqrt{A(A + Ckk + Ek^4)} + k\sqrt{A(A + Cyy + Ey^4)}}{A - Ekkyy};$$

quae cum constantem novam k involvat ab arbitrio nostro
revera integralis completa.

1) Editio princeps: $\cos. \Phi = \frac{81u + 36st - 8s^3}{8(ss - 3t)\sqrt{(ss + 3t)}}$. Correxit A. K.

hoc casu ponamus

$$\int \frac{dx}{\sqrt{(A + Cxx + Ex^4)}} = \Pi . x,$$

0 fit $x = k$, erit $\Pi . x = \Pi . k + \Pi . y$. Hinc, si fiat $k = y$,

$$x = \frac{2y\sqrt{A(A + Cyy + Ey^4)}}{A - Ey^4},$$

y ideoque iste valor ipsius x satisfacit huic aequationi diffe-

$$\frac{dx}{\sqrt{(A + Cxx + Ex^4)}} = \frac{2dy}{\sqrt{(A + Cyy + Ey^4)}};$$

m constantem non complectitur, erit is tantum integrale in-

tamen et huius aequationis differentialis facile integrale
peri poterit. Ponatur enim

$$\frac{dy}{\sqrt{(A + Cyy + Ey^4)}} = \frac{dz}{\sqrt{(A + Czz + Ez^4)}}$$

$$y = \frac{z\sqrt{A(A + Ckk + Ek^4)} + k\sqrt{A(A + Czz + Ez^4)}}{A - Ekkzz},$$

substituatur in formula

$$x = \frac{2y\sqrt{A(A + Cyy + Ey^4)}}{A - Ey^4},$$

r x per z et novam constantem arbitrariam k, qui valor erit
um huius aequationis differentialis

$$\frac{dx}{\sqrt{(A + Cxx + Ex^4)}} = \frac{2dz}{\sqrt{(A + Czz + Ez^4)}}.$$

nus $\Pi . k = n\Pi . y$ ac sumamus valorem ipsius k iam esse
ex praecedentibus colligimus, si capiatur

$$x = \frac{y\sqrt{A(A + Ckk + Ek^4)} + k\sqrt{A(A + Cyy + Ey^4)}}{A - Ekkyy},$$

fore $\Pi.x = (n+1)\Pi.y$. Cum igitur casu $n = 1$ sit
x inventus dabit valorem ipsius k pro casu $n = 2$, u
$\Pi.x = 3\Pi.y$. Qui valor porro pro k sumtus cum pr
x, ut fiat $\Pi.x = 4\Pi.y$, sicque, quousque lubuerit, pro

34. Invento autem valore ipsius x, ut sit $\Pi.x =$
particulare huius aequationis differentialis

$$\frac{dx}{\sqrt{(A + Cxx + Ex^4)}} = \frac{ndy}{\sqrt{(A + Cyy + Ey}}$$

tum vero capiatur

$$z = \frac{x\sqrt{A(A + Ckk + Ek^4)} + k\sqrt{A(A + Cxx}}{A - Ekkxx}$$

sicque obtinebitur valor integralis ipsius z complet
differentiali

$$\frac{dz}{\sqrt{(A + Czz + Ez^4)}} = \frac{ndy}{\sqrt{(A + Cyy + Ey}}$$

erit enim $\Pi.z = \Pi.k + \Pi.x = \Pi.k + n\Pi.y$.

35. Contemplemur nunc etiam in genere form
eamque ad lineam curvam $akfgpqrst$ (Fig. 1) transf

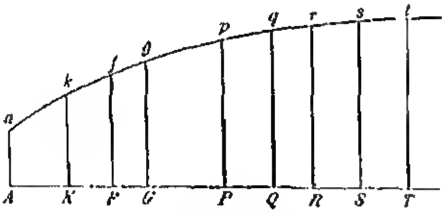


Fig. 1.

indoles, ut posit
$AK = x$ arcus ipsi

$$ak = \int \frac{dx(}{\sqrt{(}}$$

quem hoc signo
festum autem est
relatio inter arcum ak et suam abscissam AK est st
inter arcum et applicatam vel cordam aliamve rectam,
licet, constitui potuisse. Quare etsi hic x abscissam
designat, tamen quoque aliam quamvis rectam ad arcu
poterit, dummodo ea evanescat ipso arcu evanescente.

nsideremus nunc ternas abscissas, quae sint $AK = k$, $AF = f$ et
ae ita a se invicem pendeant, ut sit

$$g = \frac{f\sqrt{A(A + Ckk + Ek^4)} + k\sqrt{A(A + Cff + Ef^4)}}{A - Ekkff},$$

$$f = \frac{g\sqrt{A(A + Ckk + Ek^4)} - k\sqrt{A(A + Cgg + Eg^4)}}{A - Ekkgg},$$

$$k = \frac{g\sqrt{A(A + Cff + Ef^4)} - f\sqrt{A(A + Cgg + Eg^4)}}{A - Effgg},$$

arcus $ak = \Pi.k$, $af = \Pi.f$ et $ag = \Pi.g$ haec relatio locum
sit

$$\Pi.f - \Pi.k = \text{Arc.}\, ag - \text{Arc.}\, af - \text{Arc.}\, ak = \text{Arc.}\, fg - \text{Arc.}\, ak$$
$$= \frac{\mathfrak{B}kfg}{\sqrt{A}} + \frac{\mathfrak{C}kfg(kk + ff + gg)}{2\sqrt{A}} - \frac{\mathfrak{C}Ek^3f^3g^3}{6A\sqrt{A}}.$$

o ergo quocunque arcu ak a curvae initio a sumto a quovis puncto
oterit arcus fg, ita ut differentia arcuum fg et ak geometrice
ont. Ob puncta enim k et f data dabuntur abscissae k et f, ex
formulam primam definitur abscissa g. Vel etiam, si dentur
g, a puncto g regrediendo abscindi poterit arcus gf, qui ab arcu
o geometrica discrepet. Vel denique dato arcu quocunque fg a
a abscindi poterit arcus ak, qui ab illo quantitate geometrica

us hic evolvi meretur, quo $f = k$; si igitur abscissa $AG = g$
accipiatur, ut sit

$$g = \frac{2k\sqrt{A(A + Ckk + Ek^4)}}{A - Ek^4}$$

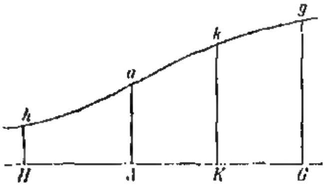


Fig. 2.

$K = k$, erit

$$\text{Arc.}\, ak = \frac{\mathfrak{B}kkg}{\sqrt{A}} + \frac{\mathfrak{C}kkg(2kk + gg)}{2\sqrt{A}} - \frac{\mathfrak{C}Ek^6g^3}{6A\sqrt{A}}.$$

fuerit $Ek^4 > A$, valor ipsius g prodibit negativus, qui ergo retro

sumtus fit $AH = h$, ita ut sit $g = -h$ et $\Pi . g = -\Pi . h$ ex

$$h = \frac{2k\sqrt{A}(A + Ckk + Ek^4)}{Ek^4 - A},$$

eritque mutatis signis

$$\text{Arc.}\, ah + 2\,\text{Arc.}\, ak = \frac{\mathfrak{B}kkh}{\sqrt{A}} + \frac{\mathfrak{C}kkh(2kk+hh)}{2\sqrt{A}} - \frac{\mathfrak{C}}{6A}$$

39. Hinc intelligitur abscissam k eiusmodi valorem obtin $h = k$; quare si curva ex puncto a utrinque per ramos si extendatur fueritque $AH = AK$, erit quoque $\text{Arc.}\, ah = \text{Arc}$ $h = k$ seu

$$Ek^4 - A = 2\sqrt{A}(A + Ckk + Ek^4)$$

vel

$$EEk^8 - 6AEk^4 - 4ACkk - 3AA = 0,$$

erit

$$3\,\text{Arc.}\, ak = \frac{\mathfrak{B}k^3}{\sqrt{A}} + \frac{3\mathfrak{C}k^5}{2\sqrt{A}} - \frac{\mathfrak{C}Ek^9}{6A\sqrt{A}};$$

arcus ergo huic abscissae $AK = k$ respondens absolute c cum sit

$$\text{Arc.}\, ak = \frac{\mathfrak{B}k^3}{3\sqrt{A}} + \frac{\mathfrak{C}k^5}{2\sqrt{A}} - \frac{\mathfrak{C}Ek^9}{18A\sqrt{A}}.$$

40. Aequatio autem illa, etsi est octavi gradus, commod positis enim eius factoribus

$$(k^4 + \alpha kk + \beta)(k^4 - \alpha kk + \gamma) = 0$$

reperitur

$$\beta + \gamma = \alpha\alpha - \frac{6A}{E}, \quad \beta - \gamma = \frac{4AC}{\alpha EE} \quad \text{et} \quad \beta\gamma = -$$

unde oritur

$$\alpha^4 - \frac{12A}{E}\alpha\alpha + \frac{36AA}{EE} - \frac{16AACC}{\alpha\alpha E^4} = - \frac{12AA}{EE}$$

hincque

$$\alpha\alpha = \frac{4A}{E} + \sqrt[3]{\frac{16AACC - 64A^3E}{E^4}}$$

et ob

$$\gamma = \frac{\alpha\alpha}{2} - \frac{3A}{E} - \frac{2AC}{\alpha EE}$$

$$kk = \frac{1}{2}\alpha \pm \sqrt{\left(\frac{2AC}{\alpha EE} + \frac{3A}{E} - \frac{1}{4}\alpha\alpha\right)}$$

$$kk = -\frac{1}{2}\alpha + \sqrt{\left(\frac{-2AC}{\alpha EE} + \frac{3A}{E} - \frac{1}{4}\alpha\alpha\right)}.$$

n quod abscissae negativae idem arcus negative sumtus respon-
stis curvis semper locum habet. Nam cum sit

$$\Pi.\,x = \int \frac{dx(\mathfrak{A} + \mathfrak{B}xx + \mathfrak{C}x^4)}{\sqrt{(A + Cxx + Ex^4)}},$$

capiatur negativa, erit

$$\Pi.(-x) = \int \frac{-dx(\mathfrak{A} + \mathfrak{B}xx + \mathfrak{C}x^4)}{\sqrt{(A + Cxx + Ex^4)}} = -\Pi.\,x.$$

quoties abscissae k in § praec. definitae respondet arcus realis,
arcus longitudinem geometrice assignari posse.

nare autem non ausim hoc ratiocinium, quo arcum absolute recti-
i, semper tuto adhiberi posse; videntur enim casus existere,
m non sit habiturum. Si enim sit $\mathfrak{B} = 0$ et $\mathfrak{C} = 0$ ideoque

$$\Pi.\,x = \int \frac{\mathfrak{A}dx}{\sqrt{(A + Cxx + Ex^4)}},$$

39 utique 3 Arc. $ak = 0$, cum tamen ex aequatione octavi gradus
non fiat abscissa $k = 0$. Verum recordandum est hanc aequa-
esse ex hac

$$k = \frac{2k\sqrt{A(A + Ckk + Ek^4)}}{Ek^4 - A};$$

tim praebeat radicem $k = 0$, haec unica erit, quae hoc casu
acit reliquis existentibus omnibus ineptis.

e tamen his casibus ratiocinium omnino fallere censendum est,
alia quaecunque radix accipiatur, sed potius eidem abscissae
respondere sunt putandi, quorum unus tantum isque negativus

satisfaciat; hocque ergo casu, tametsi in § 38 statua... non sequitur esse $\mathrm{Arc.}\,ah = \mathrm{Arc.}\,ak$ ideoque $\mathrm{Arc.}\,a$... cum eidem abscissae $h = k$ etiam alii arcus praeter A... quos unus sit, qui reddat $\mathrm{Arc.}\,ah + 2\,\mathrm{Arc.}\,ak = 0$.

44. Quod quo clarius perspiciatur, ponamus $A =$ ... stente $\mathfrak{B} = 0$ et $\mathfrak{C} = 0$ eritque $\Pi.x = \mathfrak{A}\,\mathrm{Arc.\,tang.}\,x$ ... atque $\mathrm{Arc.}\,ah = \mathfrak{A}\,\mathrm{A\,tang.}\,h$; posito ergo

$$h = \frac{2k\sqrt{(1 + 2kk + k^4)}}{k^4 - 1} = \frac{2k}{kk - 1}$$

erit $\mathfrak{A}\,\mathrm{A\,tang.}\,h + 2\mathfrak{A}\,\mathrm{A\,tang.}\,k = 0$. Quodsi iam pona... et $k = \sqrt{3}$ reperieturque $\mathfrak{A}(\mathrm{A\,tang.}\sqrt{3} + 2\,\mathrm{A\,tang.}\sqrt{3})$... $\mathrm{A\,tang.}\sqrt{3} = \mathrm{Arc.}\,60^0$, tamen inde non sequitur $3\mathfrak{A}\,\mathrm{A}$... esset falsum; sed quoniam tangenti $\sqrt{3}$ convenit qu... valor priori loco pro $\mathrm{A\,tang.}\sqrt{3}$ scriptus veritatem p...

$$\mathfrak{A}(-\mathrm{Arc.}\,120^0 + 2\,\mathrm{Arc.}\,60^0) = 0.$$

45. Haec igitur ambiguitas, qua eidem quantitati... abscissam assumimus, plures valores $\mathrm{Arc.}\,ak$ responder... quod, etiamsi in § 38 ponatur $h = k$, non tamen pro ... bere liceat $3\,\mathrm{Arc.}\,ak$. Interim tamen nihilominus erit

$$\mathrm{Arc.}\,ah + 2\,\mathrm{Arc.}\,ak = \frac{\mathfrak{B}k^3}{\sqrt{A}} + \frac{3\mathfrak{C}k^5}{2\sqrt{A}} - \frac{\mathfrak{C}}{6}\ldots$$

abscissae enim h, etsi est $= k$, tamen praeter arcum... conveniet, qui loco $\mathrm{Arc.}\,ah$ substitutus aequationi sati... guitatem sedulo dispici oportet, ne in errorem induca...

46. Quoties autem huiusmodi ambiguitas non hab... abscissae unicus arcus respondeat, tum sine haesitation... etiam pro $\mathrm{Arc.}\,ah$ scribere licebit $\mathrm{Arc.}\,ak$ et $3\,\mathrm{Arc.}\,ak$ ... neque hinc ullus error erit extimescendus, quaecu... octavi gradus § 39 inventae pro k capiatur. Id quo... quo $\mathfrak{A} = A$, $\mathfrak{B} = 2C$ et $\mathfrak{C} = 3E$, quippe quo fit

$$\Pi.x = x\sqrt{(A + Cxx + Ex^4)}$$

quantitas algebraica et

$$\Pi.g - \Pi.f - \Pi.k = \frac{2Ckfg}{\sqrt{A}} + \frac{3Ekfg(kk+ff+gg)}{2\sqrt{A}} - \frac{EEk^3f^3g^3}{2A\sqrt{A}}.$$

Quodsi iam ponatur $f=k$, erit

$$g = \frac{2k\sqrt{A}(A+Ckk+Ek^4)}{A-Ek^4}$$

$$\sqrt{A}(A+Cgg+Eg^4) = \frac{A(gg-2kk)+Ek^4gg}{2kk}.$$

$g = -k$ seu

$$Ek^4 - A = 2\sqrt{A}(A+Ckk+Ek^4);$$

$$\sqrt{A}(A+Cgg+Eg^4) = \frac{A+Ek^4}{2} = \sqrt{A}(A+Ckk+Ek^4);$$

$g = -\Pi.k$ et

$$= \frac{-2Ck^3}{\sqrt{A}} - \frac{9Ek^5}{2\sqrt{A}} + \frac{EEk^9}{2A\sqrt{A}} \quad \text{seu} \quad 3\Pi.k = \frac{k(4ACkk+9AEk^4-EEk^8)}{2A\sqrt{A}}.$$

$$EEk^8 = 6AEk^4 + 4ACkk + 3AA,$$

$$\Pi.k = \frac{k(3AEk^4-3AA)}{2A\sqrt{A}} = \frac{3k(Ek^4-A)}{2\sqrt{A}} = 3k\sqrt{(A+Ckk+Ek^4)},$$

$\Pi.k = k\sqrt{(A+Ckk+Ek^4)}$ est veritati consentaneum.

Quamquam autem haec curva per se est rectificabilis, tamen evidenter d, quod volumus, scilicet contineri in nostris formulis etiam curvas abiles, in quibus modo ante exposito arcum absolute rectificabilem liceat. Invento autem uno arcu rectificabili velut ak ex eo statim alii eiusdem indolis exhiberi poterunt; cum enim a quovis puncto li queat arcus fg, cuius ab illo differentia est geometrica, etiam hic it rectificabilis. Praeterea vero ex eodem arcu adhuc alii infiniti rectificabiles reperientur modo sequenti, quem in genere exponere

ut sit per § 36

$$g = \frac{fK + kF}{A - Ekkff}, \quad f = \frac{gK - kG}{A - Ekkgg}, \quad k = $$

Quodsi iam fuerit

$$\Pi. x = \int \frac{dx(\mathfrak{A} + \mathfrak{B}xx + \mathfrak{C}x^4)}{\sqrt{(A + Cxx + Ex^4)}},$$

erit

$$\Pi. g - \Pi. f - \Pi. k = \text{Arc. } ag - \text{Arc. } af - \text{Arc. } ak$$

$$= \frac{\mathfrak{B}kfg}{\sqrt{A}} + \frac{\mathfrak{C}kfg(kk + ff + gg)}{2\sqrt{A}} - \frac{\mathfrak{C}Ek}{6A}$$

50. Sumatur simili modo praeter abscissam AK
$AP = p$, $AQ = q$ positoque pariter

$$\sqrt{A}(A + Cpp + Ep^4) = P \quad \text{et} \quad \sqrt{A}(A + C$$

ac relatione hac constituta

$$q = \frac{pK + kP}{A - Ekkpp}, \quad p = \frac{qK - kQ}{A - Ekkqq}, \quad k =$$

erit pro eadem curva

$$\text{Arc. } pq - \text{Arc. } ak = \frac{\mathfrak{B}kpq}{\sqrt{A}} + \frac{\mathfrak{C}kpq(kk + pp + q}{2\sqrt{A}}$$

51. Subtracta ergo illa aequatione ab hac relin

$$\text{Arc. } pq - \text{Arc. } fg$$

$$= \frac{\mathfrak{B}k(pq - fg)}{\sqrt{A}} + \frac{\mathfrak{C}kpq(kk + pp + qq) - \mathfrak{C}kfg(kk + ff +}{2\sqrt{A}}$$

ubi abscissae f, g, p et q ita a se invicem pendent,

$$k = \frac{gF - fG}{A - Effgg} = \frac{qP - pQ}{A - Eppqq} \quad \text{vel} \quad \frac{1}{k} = \frac{gF +}{A(gg}$$

unde simul abscissa k eliminari et relatio inter f, g

haec eliminatio facilius absolvatur, notandum est esse quoque

$$\frac{A(ff+gg-kk)-Ekkffgg}{2fg} = \frac{A(pp+qq-kk)-Ekkppqq}{2pq},$$

$$\frac{Apq(ff+gg)-Afg(pp+qq)}{(pq-fg)(A-Efgpq)} = \frac{(gF-fG)^2}{(A-Effgg)^2} = \frac{(qP-pQ)^2}{(A-Eppqq)^2}.$$

$$pp+qq)-fg(kk+ff+gg) = pq(pp+qq)-fg(ff+gg) + \frac{Apq(ff+gg)-Afg(pp+qq)}{A-Efgpq}$$

etur

$$q - \text{Arc.}\, fg = \frac{\mathfrak{B}k(pq-fg)}{\sqrt{A}} + \frac{\mathfrak{C}k(pq-fg)(ff+gg+pp+qq)}{2\sqrt{A}} - \frac{\mathfrak{C}Ek(pq-fg)^2(pq(ff+gg)-fg(pp+qq))}{6(A-Efgpq)\sqrt{A}}.$$

igitur sit

$$kk = \frac{A(pq(ff+gg)-fg(pp+qq))}{(pq-fg)(A-Efgpq)}$$

scissae f, g, p, q ita a se invicem pendeant, ut sit

$$\frac{gF+fG}{gg-ff} = \frac{qP+pQ}{qq-pp},$$

lo arcu quocunque fg in curva assumta semper ab alio dato
cindi posse arcum pq, qui ab illo arcu differat quantitate alge-
bili.

lsi porro a puncto q ulterius progrediendo capiatur punctum r,
abscissa $AR = r$ sit

$$\frac{gF+fG}{gg-ff} = \frac{rQ+qR}{rr-qq}$$

$$\frac{fg(pp+qq)}{1-Efgpq)} = \frac{qr(ff+gg)-fg(qq+rr)}{(qr-fg)(A-Efgqr)} = \frac{qr(pp+qq)-pq(qq+rr)}{(qr-pq)(A-Epqqr)},$$

erit quoque Arc. gr — Arc. fg = quantitati algebraicae
priorem addita dabit

$$\text{Arc.}\,pr - 2\,\text{Arc.}\,fg = \text{Quant. algebr}$$

sicque a dato puncto p abscindi potest arcus pr, qui
situm fg superet quantitate algebraica.

55. Simili modo, si ulterius abscissae $AS = s$, A
ut sit

$$\frac{gF + fG}{gg - ff} = \frac{sR + rS}{ss - rr} = \frac{tS + sT}{tt - ss} \quad \text{et}$$

arcus ps triplum arcus fg, arcus pt quadruplum a
quantitate geometrice assignabili. Vicissim autem da
vel pt etc. reperiri poterit a dato puncto f arcus fg,
vel triente vel quadrante deficiat quantitate geometric

56. Evenire etiam posset, ut, licet quantitates 𝔄
aequales, tamen differentiae istae geometrice assigna
etiam semper una abscissarum ita definiri potest, ut
in nihilum abeat. His igitur casibus in proposita cu
assignari poterunt, qui inter se vel aequales sint fut
numeri ad numerum habituri.

57. Cum haec latissime pateant atque ad omne
queant, quarum arcus pro abscissa vel alia quacunqu
exprimitur, ut sit

$$= \int \frac{dx(\mathfrak{A} + \mathfrak{B}xx + \mathfrak{C}x^4)}{\sqrt{(A + Cxx + Ex^4)}},$$

conveniet istas affectiones pro nonnullis curvis determ
huius methodi clarius perspiciatur. Primum igitur po
comparationem in ellipsi exponere visum est.

DE COMPARATIONE ARCUUM IN E

58.[1]) Sit igitur propositus quadrans ellipticus A
centrum in A; ponatur alter semiaxis, super qu

1) In editione principe paragraphorum numeri abhinc desunt.

ter vero $Aa = na$. Sumta ergo abscissa quacunque $AP = x$ erit

$$PM = n\sqrt{(aa - xx)}$$

rentiale

$$= -\frac{nxdx}{\sqrt{(aa-xx)}},$$

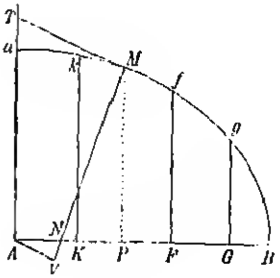


Fig. 3.

us huic abscissae respondens

$$aM = \int dx \sqrt{\frac{aa + (nn-1)xx}{aa - xx}}.$$

$- nn = m$, ut sit

$$aM = \int dx \sqrt{\frac{aa - mxx}{aa - xx}}.$$

e est, uter semiaxium sit maior vel minor, sumamus AB esse eoque $n < 1$ et m numerus positivus unitate minor, et cum focus miaxe AB, erit eius a centro A distantia

$$= \sqrt{(aa - nnaa)} = a\sqrt{m};$$

numeri m facilius intelligitur.

ergo arcus abscissae cuicunque $AP = x$ respondens designetur erit

$$\Pi.\, x = \int dx \sqrt{\frac{aa - mxx}{aa - xx}},$$

io ad formam nostram generalem reducta abibit in hanc

$$\Pi.\, x = \int \frac{dx(aa - mxx)}{\sqrt{(a^4 - (m+1)aaxx + mx^4)}}.$$

oc casu habebimus istos valores

$$C = -(m+1)aa, \quad E = m, \quad \mathfrak{A} = aa, \quad \mathfrak{B} = -m \quad \text{et} \quad \mathfrak{C} = 0.$$

tribus abscissis k, x, y, quibus respondeant arcus $\Pi.\, k$, $\Pi.\, x$, sit

$$\frac{aay\sqrt{(a^4 - (m+1)aakk + mk^4)} + aak\sqrt{(a^4 - (m+1)aayy + my^4)}}{a^4 - mkkyy},$$

$$\frac{aax\sqrt{(a^4 - (m+1)aakk + mk^4)} - aak\sqrt{(a^4 - (m+1)aaxx + mx^4)}}{a^4 - mkkxx},$$

$$\frac{aax\sqrt{(a^4 - (m+1)aayy + my^4)} - aay\sqrt{(a^4 - (m+1)aaxx + mx^4)}}{a^4 - mxxyy},$$

hi tres arcus a se invicem ita pendebunt, ut sit

$$\Pi . x - \Pi . y - \Pi . k = -\frac{mkxy}{aa}.$$

His igitur praemissis sequentia problemata resolvamus.

PROBLEMA 1

59. *Proposito ellipseos arcu quocunque ak* (Fig. 3, p. *puncto f abscindere arcum fg, ita ut differentia arcuum a assignari queat.*

SOLUTIO

Ductis ex punctis k, f, g applicatis kK, fF, gG $AK = k$, $AF = f$, $AG = g$, quarum illae dantur, hae eruntque arcus

$$ak = \Pi . k, \quad af = \Pi . f, \quad ag = \Pi . g.$$

Ponatur porro brevitatis gratia secundum § 49

$$aa\sqrt{(a^4 - (m+1)aakk + mk^4)} = K,$$

$$aa\sqrt{(a^4 - (m+1)aaff + mf^4)} = F,$$

$$aa\sqrt{(a^4 - (m+1)aagg + mg^4)} = G$$

ac statuatur inter ternas abscissas ista relatio

$$g = \frac{fK + kF}{a^4 - mkkff} \quad \text{vel} \quad f = \frac{gK - kG}{a^4 - mkkgg} \quad \text{vel} \quad k = \frac{g}{a^4}$$

quo facto habebitur

$$\Pi . g - \Pi . f - \Pi . k = \text{Arc. } fg - \text{Arc. } ak = -$$

Puncto g ergo ita sumto, ut sit

$$AG = g = \frac{fK + kF}{a^4 - mkkff},$$

differentia arcuum ak et fg geometrice poterit assignari.

$$\text{Arc. } ak - \text{Arc. } fg = \frac{mkfg}{aa}.$$

Q. E. I.

COROLLARIUM 1

60. Eadem solutio locum habebit, si proposito arcu ak detur punctum
o regrediendo versus a abscindi oporteat arcum gf, cuius ab illo differe
at esse geometrica; tum enim abscissae k et g erunt datae, ex qui
tertiae f reperiri poterit.

COROLLARIUM 2

61. Dato etiam arcu quocumque in ellipsi fg a vertice a abscindi pot
s ak, ita ut differentia arcuum ak et fg fiat geometrica. Ita cuius
s fg rectificatio pendebit a rectificatione arcus cuiusdam ak in ver
sis a terminato.

COROLLARIUM 3

62. Relatio inter ternas abscissas k, f, g etiam ita exhiberi potest, ut

$$g = \frac{a^4(-kk+ff)}{fK-kF} \quad \text{vel} \quad f = \frac{a^4(-kk+gg)}{gK+kG} \quad \text{vel} \quad k = \frac{a^4(gg-ff)}{gF+fG},$$

uibus cum praecedentibus comparatis elicitur

$$K = \frac{a^4(ff+gg-kk)-mkkffgg}{2fg} = aa\sqrt{(aa-kk)(aa-mkk)},$$

$$F = \frac{a^4(kk+gg-ff)-mkkffgg}{2kg} = aa\sqrt{(aa-ff)(aa-mff)},$$

$$G = \frac{-a^4(kk+ff-gg)+mkkffgg}{2kf} = aa\sqrt{(aa-gg)(aa-mgg)};$$

vero etiam habebitur

$$fg(gg-ff)K-kg(gg-kk)F-kf(ff-kk)G=0.$$

COROLLARIUM 4

63. Si differentia inter arcus ak et fg omnino debeat evanescere, p
eri non posse, nisi sit vel $k=0$ vel $f=0$ vel $g=0$. Primo casu
s ak ideoque et arcus fg evanescit, binis reliquis casibus autem alter
minus arcus fg in punctum a incidit fitque arcus fg arcui ak non so
ualis, sed etiam similis.

64. Quo ista abscissarum relatio facilius ad praxin t
iuvabit in genere, si ad punctum M ducatur normalis
perpendiculum demittatur AV, quod parallelum erit
ponatur $AP=x$, fore

$$PM=n\sqrt{(aa-xx)},\quad PN=nnx,\quad AN=mx,\quad M$$

$$AV=\frac{mx\sqrt{(aa-xx)}}{\sqrt{(aa-mxx)}},\quad NV=\frac{mnxx}{\sqrt{(aa-mxx)}},\quad M$$

$$MT=\frac{x\sqrt{(aa-mxx)}}{\sqrt{(aa-xx)}},\quad AT=\frac{naa}{\sqrt{(aa-xx)}}\quad \text{et}$$

COROLLARIUM 6

65. Posito ergo g pro x isti valores pro puncto

$$g=\frac{a^2k\sqrt{(aa-ff)(aa-mff)}+aaf\sqrt{(aa-kk)}}{a^4-mkkff}$$

$$\sqrt{(aa-gg)}=\frac{a^3\sqrt{(aa-kk)(aa-ff)}-akf\sqrt{(aa-}}{a^4-mkkff}$$

$$\sqrt{(aa-mgg)}=\frac{a^3\sqrt{(aa-mkk)(aa-mff)}-makf\sqrt{(}}{a^4-mkkff}$$

atque

$$\sqrt{(aa-gg)(aa-mgg)}$$

$$=\frac{a^4kf(2maakk+ff)-(m+1)(a^4+mkkff))+aa(a^4+mkkff)\sqrt{(aa-k}}{(a^4-mkkff)^2}$$

unde porro elicitur

$$aa\sqrt{(aa-mgg)}+mkf\sqrt{(aa-gg)}=a\sqrt{(aa-}$$

$$aa\sqrt{(aa-gg)}+kf\sqrt{(aa-mgg)}=a\sqrt{(aa-}$$

CASUS 1

66. *Proposito ellipseos arcu ak* (Fig. 4, p. 177) *in*
ab altero vertice B abscindere arcum Bf, ita ut arcuum
geometrica.

ma ergo ad hunc casum transfertur, si punctum g in vertice B eu fiat $g=a$, et quaeri oportet punctum f seu abscissam $AF=f$. $g=a$ erit $G=0$ ideoque habebitur

$$f=\frac{aK}{a^4-maakk}=a\sqrt{\frac{aa-kk}{aa-mkk}}$$

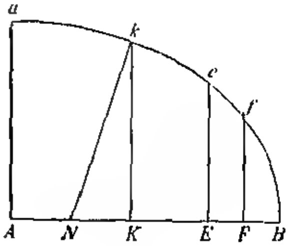


Fig. 4.

ad punctum k normali kN capi debet

$$AF=f=\frac{AB\cdot Kk}{Nk}.$$

puncto ita sumto erit arcuum differentia

$$\text{Arc. } ak-\text{Arc. } Bf=\frac{mkf}{a}=mk\sqrt{\frac{aa-kk}{aa-mkk}}=\frac{AN\cdot Kk}{Nk}.$$

COROLLARIUM

ori igitur potest, ut puncta k et f in uno puncto e coeant sicque eB in duas partes dissecetur, quarum differentia sit geometrica. tuatur $k=f=AE=e$ eritque

$$e=a\sqrt{\frac{aa-ee}{aa-mee}}\quad\text{seu}\quad a^4-2aaee+me^4=0,$$

$$ee=\frac{aa\pm aa\sqrt{(1-m)}}{m}=\frac{aa(1\pm n)}{m}$$

$-nn$. Hinc ergo erit

$$e=\frac{a}{\sqrt{(1\pm n)}}.$$

a esse debet $e<a$, erit

$$e=\frac{a}{\sqrt{(1+n)}}$$

$$AE=\frac{AB^2}{\sqrt{(AB^2+AB\cdot Aa)}}\quad\text{et}\quad Ee=\frac{na\sqrt{n}}{\sqrt{(1+n)}},$$

$$AE:Ee=1:n\sqrt{n}=AB\sqrt{AB}:Aa\sqrt{Aa}.$$

u erit

$$\text{Arc. } ae-\text{Arc. } Be=a(1-n)=AB-Aa.$$

CASUS 2

68. *Proposito arcu ak (Fig. 5) in vertice a terminato ab eius abscindere arcum kg, ita ut arcuum ak et kg differentia sit rectif*

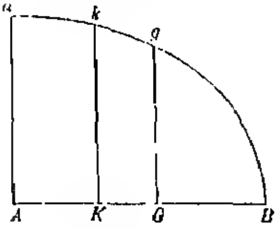


Fig. 5

Hoc ergo casu punctum f in k $f=k$ hincque etiam $F=K$; unde repe

$$AG = g = \frac{2kK}{a^4 - mk^4} = \frac{2aak\sqrt{(aa-kk)}}{a^4 - m}$$

Sumta ergo abscissa AG huius valoris

$$\text{Arc. } ak - \text{Arc. } kg = \frac{mkkg}{aa} = \frac{2mk^3\sqrt{(aa-kk)(aa-mk}}{a^4 - mk^4}$$

COROLLARIUM 1

69. Vicissim ergo arcus quicunque ag in vertice a termina duas partes secari poterit, ut partium differentia $ak - kg$ fia Ob cognitam enim abscissam $AG = g$ abscissa quaesita A aequatione definiri debet

$$gg(a^4 - mk^4)^2 = 4a^4kk(aa-kk)(aa-mkk),$$

quae abit in hanc octavi gradus

$$mmggk^8 - 4ma^4k^6 - 2ma^4ggk^4 + 4(m+1)a^6k^4 - 4a^8kk + a$$

COROLLARIUM 2

70. At si huius aequationis factores ponantur

$$(mgk^4 - Akk + a^4g)(mgk^4 - Bkk + a^4g) = 0,$$

reperitur

$$A + B = \frac{4a^4}{g} \quad \text{et} \quad AB = 4(m+1)a^6 - 4ma^4gg,$$

unde

$$A - B = \frac{4aa}{g}\sqrt{(a^4 - (m+1)aagg + mg^4)},$$

ita ut sit

$$A = \frac{2a^4 + 2aa\sqrt{(aa-gg)(aa-mgg)}}{g}$$

et

$$B = \frac{2a^4 - 2aa\sqrt{(aa-gg)(aa-mgg)}}{g}.$$

$$k^4 = \frac{2a^4kk \pm 2aakk\sqrt{(aa-gg)(aa-mgg)} - a^4gg}{mgg}$$

$$\frac{\ldots\sqrt{(aa-gg)(aa-mgg)} \pm a^3\sqrt{\left(2aa-(m+1)gg \pm 2\sqrt{(aa-gg)(aa-mgg)}\right)}}{mgg}.$$

COROLLARIUM 3

ernae ergo radices ipsius kk sunt

$$\frac{a^4 + aa\sqrt{(aa-gg)(aa-mgg)} + a^3\sqrt{(aa-gg)} + a^3\sqrt{(aa-mgg)}}{mgg},$$

$$\frac{a^4 + aa\sqrt{(aa-gg)(aa-mgg)} - a^3\sqrt{(aa-gg)} - a^3\sqrt{(aa-mgg)}}{mgg},$$

$$\frac{a^4 - aa\sqrt{(aa-gg)(aa-mgg)} + a^3\sqrt{(aa-gg)} - a^3\sqrt{(aa-mgg)}}{mgg},$$

$$\frac{a^4 - aa\sqrt{(aa-gg)(aa-mgg)} - a^3\sqrt{(aa-gg)} + a^3\sqrt{(aa-mgg)}}{mgg},$$

ambiguitate hoc modo coniunctim repraesentari possunt

$$kk = \frac{aa}{mgg}\left(a \pm \sqrt{(aa-gg)}\right)\left(a \pm \sqrt{(aa-mgg)}\right).$$

COROLLARIUM 4

autem valores ipsius k erunt hinc

$$\pm \frac{a}{g\sqrt{m}}\left(\sqrt{\frac{a+g}{2}} \pm \sqrt{\frac{a-g}{2}}\right)\left(\sqrt{\frac{a+g\sqrt{m}}{2}} \pm \sqrt{\frac{a-g\sqrt{m}}{2}}\right),$$

ino numero octo, quaterni affirmativi totidemque negativi illis-; manifestum autem est affirmativos tantum hic locum habere, qui praebent $k < g$. Hic autem est certo

$$= \frac{a}{g\sqrt{m}}\left(\sqrt{\frac{a+g}{2}} - \sqrt{\frac{a-g}{2}}\right)\left(\sqrt{\frac{a+g\sqrt{m}}{2}} - \sqrt{\frac{a-g\sqrt{m}}{2}}\right).$$

$$\sqrt{\frac{a+g}{2}} + \sqrt{\frac{a-g}{2}} > \sqrt{a}, \quad \sqrt{\frac{a+g}{2}} - \sqrt{\frac{a-g}{2}} < \sqrt{g},$$

$$\ldots\sqrt{m} + \sqrt{\frac{a-g\sqrt{m}}{2}} > \sqrt{a}, \quad \sqrt{\frac{a+g\sqrt{m}}{2}} - \sqrt{\frac{a-g\sqrt{m}}{2}} < \sqrt{g\sqrt{m}}.$$

COROLLARIUM 5

73. Si ponatur

$$\frac{y}{a} = \cos.\eta \quad \text{et} \quad \frac{y\sqrt{m}}{a} = \cos.\theta,$$

ob $m<1$ erit $\theta>\eta$ et formula nostra pro radicibus
hanc abibit formam

$$k = \pm \frac{a}{\cos.\theta}\left(\cos.\frac{1}{2}\eta \pm \sin.\frac{1}{2}\eta\right)\left(\cos.\frac{1}{2}\theta \pm\right.$$

seu ob

$$\cos.\theta = \cos.\frac{1}{2}\theta^2 - \sin.\frac{1}{2}\theta^2$$

habebitur

$$k = \pm a \cdot \frac{\cos.\frac{1}{2}\eta \pm \sin.\frac{1}{2}\eta}{\cos.\frac{1}{2}\theta \pm \sin.\frac{1}{2}\theta}.$$

Vel octoni valores erunt

$$k = \pm a \cdot \frac{\cos.\left(45^0 - \frac{1}{2}\eta\right)}{\cos.\left(45^0 - \frac{1}{2}\theta\right)}, \qquad k = \pm a \cdot \frac{\sin.\left(\right.}{\cos.\left(\right.}$$

$$k = \pm a \cdot \frac{\cos.\left(45^0 - \frac{1}{2}\eta\right)}{\sin.\left(45^0 - \frac{1}{2}\theta\right)}, \qquad k = \pm a \cdot \frac{\sin.\left(\right.}{\sin.\left(\right.}$$

COROLLARIUM 6

74. Ex his valoribus secundus

$$k = a \cdot \frac{\sin.\left(45^0 - \frac{1}{2}\eta\right)}{\cos.\left(45^0 - \frac{1}{2}\theta\right)} = a \cdot \frac{\sin.\left(45^0 - \frac{1}{2}\right.}{\sin.\left(45^0 + \frac{1}{2}\right.}$$

semper satisfacit; fit enim, uti manifestum est, non so
$k<y$ seu $k<a\cos.\eta$. Ex primo quidem valore

$$k = a \cdot \frac{\sin.\left(45^0 + \frac{1}{2}\eta\right)}{\sin.\left(45^0 + \frac{1}{2}\theta\right)}$$

$< \theta$ ob $\eta < \theta$; verum ut sit $k < g$, oportet esse

$$\frac{\ldots \frac{1}{2}\eta)}{\ldots \frac{1}{2}\theta)} < \cos.\eta = \sin.(90^0 - \eta) = 2\sin.\left(45^0 - \frac{1}{2}\eta\right)\sin.\left(45^0 + \frac{1}{2}\eta\right)$$

$$1 < 2\sin.\left(45^0 - \frac{1}{2}\eta\right)\sin.\left(45^0 + \frac{1}{2}\theta\right)$$

$$1 < \cos.\frac{1}{2}(\theta + \eta) - \cos.\left(90^0 + \frac{1}{2}(\theta - \eta)\right)$$

$$1 < \cos.\frac{1}{2}(\theta + \eta) + \sin.\frac{1}{2}(\theta - \eta).$$

PROBLEMA 2

…posito ellipseos arcu quocunque fg (Fig. 6) *a dato puncto p abscindere … pq, ita ut horum arcuum differentia $fg - pq$ fiat geometrica.*

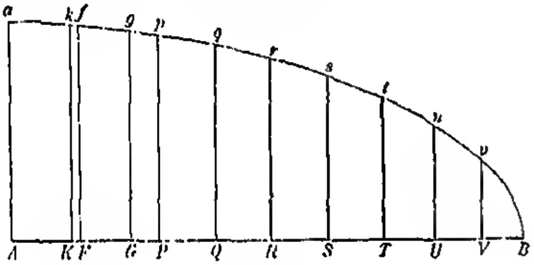


Fig. 6.

SOLUTIO

… applicatis fF, gG, pP, qQ sint abscissae $AF = f$, $AG = g$, … $AQ = q$, tum a vertice a capiatur arcus ak, qui datum arcum …to geometrica superet; positaque abscissa $AK = k$ ac brevitatis

$$K = aa\sqrt{(aa - kk)(aa - mkk)},$$

$$\ldots = aa\sqrt{(aa - ff)(aa - mff)}, \quad G = aa\sqrt{(aa - gg)(aa - mgg)},$$

$$\ldots aa\sqrt{(aa - pp)(aa - mpp)} \quad \text{et} \quad Q = aa\sqrt{(aa - qq)(aa - mqq)}$$

$$k = \frac{gF - fG}{a^4 - mffgg} = \frac{a^4(gg - ff)}{gF + fG};$$

unde reperiatur k, ita ut sit

$$\text{Arc.}\, ak - \text{Arc.}\, fg = \frac{mkfg}{aa}.$$

Tum vero abscissa q per problema praec. ita determi

$$q = \frac{pK + kP}{a^4 - mkkpp} = \frac{a^4(pp-kk)}{pK - kP},$$

eritque

$$\text{Arc.}\, ak - \text{Arc.}\, pq = \frac{mkpq}{aa},$$

a qua aequatione illa subtrahatur; relinquetur

$$\text{Arc.}\, fg - \text{Arc.}\, pq = \frac{mk}{aa}(pq - f$$

Q. E. I.

COROLLARIUM 1

76. Cum k ab abscissis p et q pari modo pende

$$k = \frac{qP - pQ}{a^4 - mppqq} = \frac{a^4(qq-pp)}{qP + pQ}$$

ideoque abscissa q ex datis f, g et p per hanc aequa

$$\frac{gF - fG}{a^4 - mffgg} = \frac{qP - pQ}{a^4 - mppqq}$$

vel etiam ex hac

$$\frac{gg - ff}{gF + fG} = \frac{qq - pp}{qP + pQ};$$

atque hinc elicitur

$$q = \frac{Fgp(pp-gg) + Gfp(pp-ff) - Pfg(}{Ff(pp-gg) + Gg(pp-ff) - Pp(gg}$$

COROLLARIUM 2

77. Abscissae p et q etiam ita ab abscissa k pe

$$aa\sqrt{(aa-mqq)} + mkp\sqrt{(aa-qq)} = a\sqrt{(aa-$$

$$aa\sqrt{(aa-qq)} + kp\sqrt{(aa-mqq)} = a\sqrt{(aa-$$

$$aa\sqrt{(aa-mpp)} - mkq\sqrt{(aa-pp)} = a\sqrt{(aa-$$

$$aa\sqrt{(aa-pp)} - kq\sqrt{(aa-mpp)} = a\sqrt{(aa-$$

$$aa\sqrt{(aa-mkk)} - mpq\sqrt{(aa-kk)} = a\sqrt{(aa-$$

$$aa\sqrt{(aa-kk)} - pq\sqrt{(aa-mkk)} = a\sqrt{(aa-$$

COROLLARIUM 3

arcuum fg et pq differentia debeat evanescere, necesse est, ut sit vel $pq = fg$. At si $k = 0$, ob

$$k = \frac{a^4(gg - ff)}{gF + fG} = \frac{a^4(qq - pp)}{qP + pQ}$$

fg quam pq evanescit. Sin autem sit $pq = fg$, ob

$$\sqrt{}(aa - mkk) - mpq\sqrt{}(aa - kk) = a\sqrt{}(aa - mpp)(aa - mqq),$$

$$\sqrt{}(aa - mkk) - mfg\sqrt{}(aa - kk) = a\sqrt{}(aa - mff)(aa - mgg)$$

$$(aa - mpp)(aa - mqq) = (aa - mff)(aa - mgg)$$

$$a\sqrt{}(aa - kk) - pq\sqrt{}(aa - mkk) = a\sqrt{}(aa - pp)(aa - qq),$$

$$a\sqrt{}(aa - kk) - fg\sqrt{}(aa - mkk) = a\sqrt{}(aa - ff)(aa - gg)$$

$$(aa - pp)(aa - qq) = (aa - ff)(aa - gg),$$

esse vel $q = g$ et $p = f$ vel $q = f$ et $p = g$; utroque autem casu q non solum aequalis, sed etiam similis arcui fg.

COROLLARIUM 4

fieri posset, ut arcus pq evanesceret manente arcu fg finito, hi rectificabilis. At evanescente arcu pq ob $q = p$ oritur $k = 0$ ideoque g; unde quoque arcus fg evanescit.

COROLLARIUM 5

arcus pq in altero vertice B debeat esse terminatus, ut sit $q = a$ hanc aequationem

$$a^2\sqrt{}(1 - m) = \sqrt{}(aa - mkk)(aa - mpp)$$

$$a^4 - aakk - aapp + mkkpp = 0 \quad \text{et} \quad kk = \frac{aa(aa - pp)}{aa - mpp}.$$

substitutus in hac aequatione

$$aa\sqrt{}(aa - kk) - fg\sqrt{}(aa - mkk) = a\sqrt{}(aa - ff)(aa - gg)$$

praebet

$$0 = a^6 + 2(1-m)a^3fgp - a^4(ff+gg$$

$$+ maa(ffgg + ffpp + ggpp) - mff$$

unde oritur

$$p = \frac{(1-m)a^3fg \pm a\sqrt{(aa-ff)(aa-gg)(aa-m}}{a^4 - maaff - maagg + mffgg}$$

qui casus ad casum problematis primi redit, si mod
se permutentur et loco abscissarum applicatae introd

COROLLARIUM 6

81. Notari quoque meretur casus, quo punctum
mitur, ita ut arcus pq arcui fg fiat contiguus sitque

$$\text{Arc.}\, fg - \text{Arc.}\, gq = \frac{mkg}{aa}(q-f$$

ob $p = g$. Cum igitur sit quoque $P = G$, erit

$$\frac{gF+fG}{gg-ff} = \frac{qG+gQ}{qq-gg},$$

unde abscissa q determinatur. Vel sumta

$$k = \frac{gF-fG}{a^4-mffgg} = \frac{a^4(gg-ff)}{gF+fG}$$

erit

$$q = \frac{gK+kG}{a^4-mkkgg} = \frac{a^4(gg-kk)}{gK-kG}.$$

Hinc autem reperitur

$$q = \frac{gg}{f} - \frac{a^4(gg-ff)^2}{f} \cdot \frac{a^4-mg^4}{2FGfg + a^4(a^4(ff+gg)-2(m+1)}$$

vel

$$q = \frac{2FGg(a^4-mg^4) - a^4f((a^4+mg^4)^2 - 2(m+1)aagg}{a^4((a^4-mg^4)^2 - 4mffgg(aa-gg)(aa}$$

vel

$$q = \frac{2FGg(a^4-mg^4) - a^4f(mg^4-2aagg+a^4)(mg^4}{a^4(a^4-mg^4)^2 - 4ma^4ffgg(aa-gg)(aa}$$

PROBLEMA 3

...osito ellipsis arcu quocunque fg a dato puncto p abscindere arcum ...plo illius arcus fg differat quantitate geometrice assignabili.

SOLUTIO

...orum f et g abscissis $AF = f$, $AG = g$ earumque quantitatibus ... G quaeratur primum abscissa

$$AK = k = \frac{gF - fG}{a^4 - mffgg} = \frac{a^4(gg - ff)}{gF + fG},$$

$$\text{Arc. } ak - \text{Arc. } fg = \frac{mkfg}{aa}.$$

...ncti p abscissam $AP = p$ quaeratur abscissa $AQ = q$, ut sit

$$q = \frac{pK + kP}{a^4 - mkkpp} = \frac{a^4(pp - kk)}{pK - kP}$$

...litteris maiusculis K et P semper eiusmodi functiones minus... p, ut, si minuscula fuerit x, valor maiusculae respondentis

$$X = aa\sqrt{(aa - xx)(aa - mxx)};$$

$$\text{Arc. } ak - \text{Arc. } pq = \frac{mkpq}{aa},$$

...us

$$\text{Arc. } fg - \text{Arc. } pq = \frac{mk}{aa}(pq - fg).$$

... si punctum q nunc tanquam datum spectetur ex eoque quae... r, ut sit eius abscissa

$$AR = r = \frac{qK + kQ}{a^4 - mkkqq} = \frac{a^4(qq - kk)}{qK - kQ},$$

$$\text{Arc. } fg - \text{Arc. } qr = \frac{mk}{aa}(qr - fg).$$

$$2\,\text{Arc.}\, fg - \text{Arc.}\, pqr = \frac{mk}{aa}(pq + qr -$$

sicque a dato puncto p abscidimus arcum pr, qui a du
quantitate algebraica. Q. E. I.

COROLLARIUM 1

83. Cum sit

$$k = \frac{a^4(gg-ff)}{gF+fG} \quad \text{et} \quad k = \frac{a^4(qq-pp}{qP+pQ}$$

similique modo

$$k = \frac{a^4(rr-qq)}{rQ+qR},$$

habebimus has aequationes

$$\frac{gF+fG}{gg-ff} = \frac{qP+pQ}{qq-pp} = \frac{rQ+qR}{rr-qq},$$

unde ex datis abscissis f, g et p reliquae duae abscis

COROLLARIUM 2

84. Si arcus fg in ipso vertice a incipiat, ut sit

$$q = \frac{pG+gP}{a^4-mggpp} = \frac{a^4(pp-gg)}{pG-gP} \quad \text{et} \quad r = \frac{qG+gQ}{a^4-mggq}$$

Ac si praeterea punctum p in altero vertice A de
$P=0$, erit

$$q = \frac{G}{a^3-magg} = \frac{a\sqrt{(aa-gg)(aa-mg}}{aa-mgg}$$

hinc

$$aa-qq = \frac{aagg(1-m)(aa-mgg)}{(aa-mgg)^2} = \frac{(1-}{aa}$$

et

$$aa-mqq = \frac{a^4(1-m)(aa-mgg)}{(aa-mgg)^2} = \frac{(1-m)a^4}{aa-mgg}, \quad \text{unde}$$

quia applicata in partem inferiorem cadere debet, erit

$$r = \frac{a(a^4-2aagg+mg^4)}{a^4-2maagg+mg^4}.$$

COROLLARIUM 3

oc ergo casu sumto r (Fig. 7) in superiore ut posita abscissa $AG = g$ sit

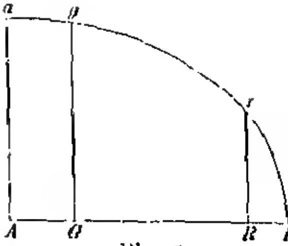


Fig. 7.

$$AR = r = \frac{a(a^4 - 2aagg + mg^4)}{a^4 - 2maagg + mg^4}$$

$$BR = a - r = \frac{2(1-m)a^3gg}{a^4 - 2maagg + mg^4},$$

c. ag — Arc. Br = Quant. algebr. $= \frac{mg}{aa}(ag + rg) = \frac{mgg}{aa}(a + r)$

$$2\,\text{Arc.}\, ag - \text{Arc.}\, Br = \frac{2mg(aa-gg)\sqrt{(aa-gg)(aa-mgg)}}{a^4 - 2maagg + mg^4}.$$

COROLLARIUM 4

puncta g et r in unum debeant coalescere, ut sit $r = g$, valo communis $AG = AR = g$ ex hac aequatione quinti gradus debe

$$mg^5 - mag^4 - 2maag^3 + 2a^3gg + a^4g - a^5 = 0.$$

$m = \frac{1}{2}$ et $a = 1$, habebitur

$$g^5 - g^4 - 2g^3 + 4gg + 2g - 2 = 0.$$

$= \frac{4}{3+\sqrt{2}}$, prodiret $g = \frac{a}{\sqrt{2}}$ foretque

$$2\,\text{Arc.}\, ag - \text{Arc.}\, Bg = a\sqrt{\frac{2+2\sqrt{2}}{3+\sqrt{2}}}.$$

PROBLEMA 4

oposito arcu ellipseos quocunque fg (Fig. 6, p. 181) *invenire arcum* pqr *cise duplo maior.*

SOLUTIO

tione ergo praecedentis problematis efficiendum est, ut sit

$$pq + qr - 2fg = 0,$$

eritque etiam 2 Arc. fg = Arc. pq. Hic autem ob arcum fg ...
praeter semiaxes $AB = a$ et $Aa = a\sqrt{(1-m)}$ dantur absci...
$AG = g$ cum valoribus derivatis F et G, unde quaeratur

$$k = \frac{a^4(gg-ff)}{gF+fG};$$

simulque erit eius valor derivatus

$$K = \frac{a^4(ff+gg-kk) - mkkffgg}{2fg}$$

(per coroll. 3. probl. 1). Simili autem modo abscissae p et ...
ut sit

$$K = \frac{a^4(pp+qq-kk) - mkkppqq}{2pq},$$

itemque ex abscissis q et r erit

$$K = \frac{a^4(qq+rr-kk) - mkkqqrr}{2qr}.$$

At ex aequatione $pq + qr = 2fg$ est $q = \frac{2fg}{p+r}$, unde obtine...
aequationes

$$K = \frac{a^4(pp-kk)(p+r)^2 + 4a^4ffgg - 4mffggkkpp ...}{4fgp(p+r)}$$

$$K = \frac{a^4(rr-kk)(p+r)^2 + 4a^4ffgg - 4mffggkkrr}{4fgr(p+r)},$$

ex quibus ambae abscissae p et r arcum quaesitum pr deter...
poterunt. Hinc ergo primum elicimus eliminando K ac per ...

$$a^4pr(p+r)^2 + a^4kk(p+r)^2 - 4a^4ffgg - 4mffggkkp ...$$

Deinde addendo illas aequationes habebimus

$$2K = \frac{a^4pr(p+r)^3 - a^4kk(p+r)^3 + 4a^4ffgg(p+r) - 4mffggk ...}{4fgpr(p+r)}$$

Ex illa autem est

$$a^4(p+r)^2 = \frac{4ffgg(a^4 + mkkpr)}{pr+kk},$$

in hac substitutus praebet

$$Kfgpr = \frac{4ffgg(pr-kk)(a^4+mkkpr)}{pr+kk} + 4a^4ffgg - 4mffggkkpr$$

$$\frac{2Kpr(pr+kk)}{fg} = 2a^4pr - 2mk^4pr;$$

itur

$$pr = \frac{(a^4-mk^4)fg - Kkk}{K} = \frac{ffgg(2a^4-mk^4) - a^4kk(ff+gg-kk)}{a^4(ff+gg-kk) - mffggkk}$$

$$(p+r)^2 = \frac{4fg}{a^4}(K+mfgkk) = \frac{2a^4(ff+gg-kk)+2mffggkk}{a^4}.$$

$$p+r = \frac{\sqrt{2(a^4(ff+gg-kk)+mffggkk)}}{aa}.$$

$$r-p = \frac{\sqrt{2(a^8(gg-ff)^2 - a^8k^4 + 2ma^4ffggk^4 - mmf^4g^4k^4)}}{aa\sqrt{(a^4(ff+gg-kk)-mffggkk)}}$$

$$r-p = \frac{\sqrt{2(a^8(gg-ff)^2 - k^4(a^4-mffgg)^2)}}{aa\sqrt{(a^4(ff+gg-kk)-mffggkk)}}.$$

m sit

$$a^4(gg-ff) = k(gF+fG) \quad \text{et} \quad a^4 - mffgg = \frac{gF-fG}{k},$$

$$r-p = \frac{2k}{aa}\sqrt{\frac{FG}{K}},$$

$$r+p = \frac{\sqrt{2(a^4(ff+gg-kk)+mffggkk)}}{aa} = \frac{2}{aa}\sqrt{fg(K+mfgkk)}$$

bscissa p et r innotescit. Q. E. I.

COROLLARIUM 1

Cum sit

$$k = \frac{gF-fG}{a^4-mffgg}$$

$$K = \frac{(a^4+mffgg)FG - a^6fg(2maa(ff+gg)-(m+1)(a^4+mffgg))}{(a^4-mffgg)^2},$$

erit

$$r + p = \frac{2}{aa}\sqrt{\frac{fgFG - ma^4ffgg(ff+gg) + (m+1)a^6ffg}{a^4 - mffgg}}$$

$$r - p = \frac{2(gF - fG)}{aa}\sqrt{\frac{FG}{(a^4 + mffgg)(FG + (m+1)a^6fg) - 2ma^8f}}$$

COROLLARIUM 2

89. Si arcus datus fg in vertice a terminetur, ut sit $f = 0$ e
$p + r = 0$ et $r - p = 2g$, unde $p = -g$ et $r = g$; arcus ergo
circa a aequaliter extenditur utrumque semissem arcui fg s
habens et aequalem. Idem evenit, si arcus datus in altero ve
netur, ut sit $g = a$ et $G = 0$; tum enim fit $r - p = 0$ et $r +$
$r = p = f$.

COROLLARIUM 3

90. Quemadmodum his casibus, ubi arcus propositus fg i
terminatur, eius arcus *duplus* per se est manifestus, ita, si ar
in neutro vertice terminatur, assignatio arcus dupli maxim
quippe qui arcus geometrice ne bisecari quidem potest.

COROLLARIUM 4

91. Hinc etiam patet, si detur vicissim arcus pr, inveniri
qui eius exacte futurus sit semissis; sed hoc non nisi molest
praestari poterit. At si arcus duplus pqr quadranti elliptico s
$p = 0$ et $r = a$, non difficulter arcus assignabitur eius semissi a
enim erit

$$q = k \quad \text{et} \quad k = a\sqrt{\frac{1 - \sqrt{(1-m)}}{m}}$$

sicque innotescit tam k quam

$$K = a^4\sqrt{\frac{1-m}{m}}\left(1 - \sqrt{(1-m)}\right).$$

Porro est

$$2fg = ak \quad \text{et} \quad ff + gg = \frac{Kk}{a^3} + kk + \frac{mk^4}{4aa}.$$

At est

$$m = \frac{2aakk - a^4}{k^4} \quad \text{ideoque} \quad ff + gg = \frac{2kk + 3aa}{4};$$

$$g + f = \tfrac{1}{2}\sqrt{(2kk + 3aa + 4ak)}$$

$$g - f = \tfrac{1}{2}\sqrt{(2kk + 3aa - 4ak)}$$

$$f = \tfrac{1}{4}\sqrt{(3aa + 4ak + 2kk)} - \tfrac{1}{4}\sqrt{(3aa - 4ak + 2kk)},$$

$$g = \tfrac{1}{4}\sqrt{(3aa + 4ak + 2kk)} + \tfrac{1}{4}\sqrt{(3aa - 4ak + 2kk)}.$$

COROLLARIUM 5

ponatur alter semiaxis $Aa = b$ existente altero $AB = a$, ut sit erit pro hoc casu $k = a\sqrt{\frac{a}{a+b}}$, quo valore substituto habebitur

$$g \pm f = \frac{a}{2}\sqrt{\left(\frac{5a+3b}{a+b} \pm 4\sqrt{\frac{a}{a+b}}\right)};$$

$$f = \frac{a}{2}\sqrt{\frac{5a+3b-\sqrt{(9aa+14ab+9bb)}}{2(a+b)}},$$

$$g = \frac{a}{2}\sqrt{\frac{5a+3b+\sqrt{(9aa+14ab+9bb)}}{2(a+b)}}.$$

ssae pro utroque termino arcus fg reperientur, qui est semissis quadrantis.

COROLLARIUM 6

c ergo casu erit

$$ff + gg = \frac{aa(5a+3b)}{4(a+b)} = aa + \frac{aa(a-b)}{4(a+b)}$$

$$fg = \frac{aa}{2}\sqrt{\frac{a}{a+b}} \quad \text{et} \quad 2fg = aa\sqrt{\frac{a}{a+b}};$$

gratia sit $a = 25$ et $b = 119$, reperietur

$$f = \frac{25}{8\sqrt{2}} \quad \text{et} \quad g = \frac{125}{4\sqrt{2}}.$$

SCHOLION

94. Hinc ergo solutionem nacti sumus istius non inelegantis

Proposito ellipsis quadrante BAa (Fig. 8) *geometrice in eo absc* fg, *qui praecise aequalis sit semissi totius arcus quadrantis* $afgB$.

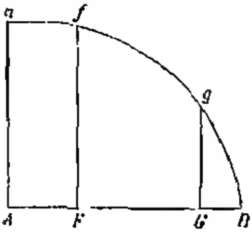


Fig. 8.

Positis enim semiaxibus $AB = a$ et punctis quaesitis f et g erunt abscissae

$$AF = \frac{a}{2}\sqrt{\frac{5a + 3b - \sqrt{(9aa + 14ab + 9}{2(a+b)}}$$

$$AG = \frac{a}{2}\sqrt{\frac{5a + 3b + \sqrt{(9aa + 14ab + 9}{2(a+b)}}$$

unde pro iisdem punctis eliciuntur applicatae

$$Ff = \frac{b}{2}\sqrt{\frac{3a + 5b + \sqrt{(9aa + 14ab + 9bb)}}{2(a+b)}},$$

$$Gg = \frac{b}{2}\sqrt{\frac{3a + 5b - \sqrt{(9aa + 14ab + 9bb)}}{2(a+b)}}.$$

PROBLEMA 5

95. *Datum ellipseos arcum* pr (Fig. 6, p. 181) *in duas partes secа ita ut differentia harum partium* $pq - qr$ *sit geometrice assignabilis.*

SOLUTIO

Positis ut in problemate praecedente $AP = p$, $AQ = q$ et stentibus semiaxibus $AB = a$ et $Aa = a\sqrt{(1-m)}$ quaeratur a vo ak, ut posita eius abscissa $AK = k$ sit

$$k = \frac{qP - pQ}{a^4 - mppqq} = \frac{a^4(qq - pp)}{qP + pQ},$$

eritque

$$\text{Arc. } ak - \text{Arc. } pq = \frac{mkpq}{aa}.$$

Tum vero sit etiam

$$k = \frac{rQ - qR}{a^4 - mqqrr} = \frac{a^4(rr - qq)}{rQ + qR};$$

$$\text{Arc. } ak - \text{Arc. } qr = \frac{mkqr}{aa}$$

$$\text{Arc. } pq - \text{Arc. } qr = \frac{mkq}{aa}(r - p).$$

dentur abscissae p et r cum suis derivatis P et R, abscissa
iti q ex hac aequatione definiri debebit

$$\frac{qP + pQ}{qq - pp} = \frac{rQ + qR}{rr - qq}$$

$$Pq(rr - qq) - Rq(qq - pp) = Q(p + r)(qq - pr),$$

o quadrata ac tum per $(qq - pp)(rr - qq)$ divisa dat

$$2qq) - 2(m + 1)aaprqq + mqq(qq(p + r)^2 - 2pprr) = 2qqPR : a^4$$

$$q^4 = \frac{2qq\left(\frac{PR}{a^4} + mpprr + (m + 1)aapr + a^4\right) - a^4(p + r)^2}{m(p + r)^2},$$

atione valor abscissae q definiri poterit. Q. E. I.

COROLLARIUM 1

otus quadrans in duas partes, quarum differentia sit geometrica,
, poni debet $p = 0$ et $r = a$; unde fit $P = a^4$ et $R = 0$ indeque

$$\frac{qq - a^4}{m} \quad \text{et} \quad qq = \frac{aa(1 - \sqrt{(1 - m)})}{m} \quad \text{et} \quad q = a\sqrt{\frac{1 - \sqrt{(1 - m)}}{m}},$$

em determinatio, quam supra iam in coroll. casus 1 probl. 1

COROLLARIUM 2

abscissarum p et r altera sit negativa alterique aequalis seu
abebitur statim vel $q = 0$ vel

$$- Rqq + Rpp = 0 \quad \text{seu} \quad qq = \frac{Prr + Rpp}{P + R} \quad \text{ideoque} \quad P + R = 0.$$

utem est, si utraque applicata Pp et Rr fuerit affirmativa, fore
tum locum habere $q = 0$.

PROBLEMA 6

98. *Si ellipsis ADBFA (Fig. 9) per diametrum quamc*
bisecta, semicircumferentiam EBF ita secare in puncto M,
FM differentia sit geometrice assignabilis.

SOLUTIO

Etsi hoc problema in praecedente continetur, tamen s
nequit, propterea quod tam $p + r = 0$ quam $P + R = 0$; p
solutio debet investigari.
axibus $CA = a$, $CD = b =$
altero termino E arcus
$CP = p$; erit applicata P
quae coordinatae negative s
terminum F pertinebunt; e
et $\frac{b}{a}\sqrt{(aa - rr)}$, ita ut
$\sqrt{(aa - rr)} = -\sqrt{(aa - pp)}$.
quadam nova abscissa k
$CQ = q$ sit ex coroll. 2 pro

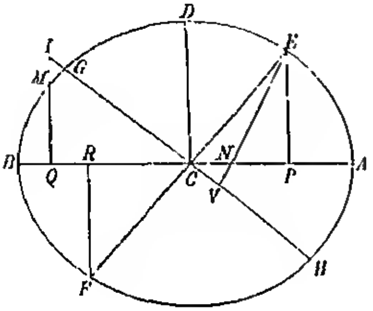


Fig. 9.

$$aa\sqrt{(aa-kk)} - pq\sqrt{(aa-mkk)} = a\sqrt{(aa-pp)(a}$$

$$aa\sqrt{(aa-kk)} - qr\sqrt{(aa-mkk)} = a\sqrt{(aa-qq)(a}$$

haec ultima aequatio ob

$$r = -p \quad \text{et} \quad \sqrt{(aa-rr)} = -\sqrt{(aa-pp}$$

abit in hanc

$$aa\sqrt{(aa-kk)} + pq\sqrt{(aa-mkk)} = -a\sqrt{(aa-pp)}$$

quae ad primam addita dat

$$2aa\sqrt{(aa-kk)} = 0 \quad \text{ideoque} \quad k = a;$$

qui valor in altera substitutus dat

$$-pq\sqrt{(1-m)} = \sqrt{(aa-pp)(aa-qq)}$$

ideoque

$$\frac{-q}{\sqrt{(aa-qq)}} = \frac{\sqrt{(aa-pp)}}{p\sqrt{(1-m)}},$$

$$q = -\frac{a\sqrt{(aa-pp)}}{\sqrt{(aa-mpp)}},$$

egativum indicat q in parte abscissarum negativa capi oportere. normalis in curvam EN; erit

$$\frac{PE}{EN} = \frac{\sqrt{(aa-pp)}}{\sqrt{(aa-mpp)}}.$$

$\frac{PE}{N}$. Sit porro GH diameter coniugata, cui normalis EN in V

$$\frac{PE}{EN} = \frac{CV}{CN} = \frac{CQ}{CI}$$

ad concursum cum applicata QM in I. Quare ob $CQ = \frac{a \cdot CQ}{CI}$ $= CA$. Unde haec sequitur constructio facilis: Diameter contra G in I continuetur, ut fiat $CI = CA$; ex I in axem AB rpendiculum IQ, quod ellipsin in puncto quaesito M secabit. erit

$$M - \text{Arc. } FM = -\frac{2mpq}{a} = \frac{2mp \cdot PE}{EN} = \frac{2CN \cdot CV}{CN} = 2CV$$

Q. E. I.

COROLLARIUM 1

iisdem aequationibus binis eliminando k problema praecedens vatur, sequens obtinebitur aequatio

$$)-p\sqrt{(aa-rr)})^2 - 2aaqq(aa+mpr)(aa-pr-\sqrt{(aa-pp)(aa-rr)})$$
$$+ a^6(\sqrt{(aa-pp)} - \sqrt{(aa-rr)})^2 = 0,$$

lutionem adipiscimur

$$a-pr-\sqrt{(aa-pp)(aa-rr)})(aa+mpr \pm \sqrt{(aa-mpp)(aa-mrr)})}{m(r\sqrt{(aa-pp)} - p\sqrt{(aa-rr)})^2},$$

$$\frac{(a-p)}{2} - \sqrt{\frac{(a-r)(a+p)}{2}}\Big)\Big(\sqrt{\frac{(a+p\sqrt{m})(a+r\sqrt{m})}{2m}} \pm \sqrt{\frac{(a-p\sqrt{m})(a-r\sqrt{m})}{2m}}\Big)}{r\sqrt{(aa-pp)} - p\sqrt{(aa-rr)}}.$$

100. Quamquam haec solutio ro a solutione pro
discrepat, tamen statim solutionem praesentis supp

$$r \quad p \quad \text{et} \quad \sqrt{(aa \quad rr)} \quad \sqrt{(a}$$

aequatio prima coroll. praec. transit in hanc forma

$$2aaqq(aa \quad mpp) \cdot 2aa \mid a^6\big(2\sqrt{(aa}$$

seu

$$qq \quad \frac{aa(aa \quad pp)}{aa \quad mpp}.$$

COROLLARIUM 3

101. Si ex duabus primis aequationibus elimin

$$q \quad \frac{aa(\sqrt{(aa \quad pp)} \quad \sqrt{(aa \quad rr)})\sqrt{(a}}{(r\sqrt{(aa \quad pp)} \quad p\sqrt{(aa \quad rr)})\sqrt{(aa}}$$

et

$$\sqrt{(aa \quad qq)} \quad \frac{a(r \quad p)\sqrt{(aa \quad k}}{r\sqrt{(aa \quad pp)} \quad p\sqrt{(aa}}$$

unde fit

$$a^4(aa \quad kk)\big(\sqrt{(aa \quad pp)} \quad \sqrt{(aa \quad rr)}\big)^2 \mid a^2(aa$$

$$aa(aa \quad mkk)\big(r\sqrt{(aa \quad pp)} \quad p\sqrt{(}$$

sive

$$mk^4(r \quad p)^2 \quad 2kk(aa \quad mpr)\big(aa \quad pr \quad \sqrt{(a}$$

$$aa\big(aa \quad pr \quad \sqrt{(aa \quad pp)}(aa$$

unde fit

$$kk \quad \frac{(aa \quad pr \quad \sqrt{(aa \quad pp)(aa \quad rr)})(aa \quad mpr \quad \sqrt{}}{m(r \quad p)^2}$$

hincque colligitur

$$k \quad \frac{\Big(\sqrt{\frac{(a+r)(a-p)}{2}} - \sqrt{\frac{(a \quad r)(a+p)}{2}}\Big)\Big(\sqrt{\frac{(a+r\sqrt{m})(a \quad p\sqrt{}}{2m}}}{r \quad p}$$

ac erit

$$aa(aa-pr-\sqrt{(aa-pp)(aa-rr)})(\sqrt{(aa-mpp)}-\sqrt{(aa-mrr)})$$
$$m(r-p)(r\sqrt{(aa-pp)}-p\sqrt{(aa-rr)})$$

rcuum pq et qr differentia sit $=\frac{mkq}{aa}(r-p)$, habebimus generaliter

$$\text{rc. } qr=\frac{(aa-pr-\sqrt{(aa-pp)(aa-rr)})(\sqrt{(aa-mpp)}-\sqrt{(aa-mrr)})}{r\sqrt{(aa-pp)}-p\sqrt{(aa-rr)}},$$

nctum q ex coroll. 1 definiatur. Erit ergo

$$-\text{Arc. } qr=\frac{(\sqrt{(aa-pp)}-\sqrt{(aa-rr)})(\sqrt{(aa-mpp)}-\sqrt{(aa-mrr)})}{r+p}$$

$$\frac{\left(\sqrt{\frac{(a+p)}{2}}-\sqrt{\frac{(a-r)(a-p)}{2}}\right)\left(\sqrt{\frac{(a+p\sqrt{m})(a+r\sqrt{m})}{2m}}-\sqrt{\frac{(a-p\sqrt{m})(a-r\sqrt{m})}{2m}}\right)}{p+r}.$$

PROBLEMA 7

roposito ellipsis arcu quocunque fg (Fig. 6, p. 181) *a dato puncto p cum pqrs, qui ab illius arcus fg triplo differat quantitate geometrice*

SOLUTIO

hactenus punctorum datorum f, g et p abscissae $AF=f$, $AG=g$, quaeratur primo arcus ak, cuius abscissa sit

$$AK=k=\frac{gF-fG}{a^4-mffgg}=\frac{a^4(gg-ff)}{gF+fG},$$

$$\text{Arc. } ak-\text{Arc. } fg=\frac{mkfg}{aa}.$$

tur punctum q, ut sit

$$AQ=q=\frac{pK+kP}{a^4-mkkpp}=\frac{a^4(pp-kk)}{pK-kP}$$

indeque

$$Q = \frac{a^4(qq-pp) - kk(a^4 - mppqq)}{2kp} = \frac{pq(qq-pp)K - kq(qq-kk)P}{kp(pp-kk)}$$

eritque

$$\text{Arc.}\, fg - \text{Arc.}\, pq = \frac{mk}{aa}(pq - fg).$$

Simili modo porro quaeratur punctum r, ut sit

$$AR = r = \frac{qK + kQ}{aa - mkkqq} = \frac{a^4(qq-kk)}{qK - kQ}$$

et

$$R = \frac{a^4(rr-qq) - kk(a^4 - mqqrr)}{2kq} = \frac{qr(rr-qq)K - kr(rr-kk)Q}{kq(qq-kk)}$$

et cum sit

$$\text{Arc.}\, fg - \text{Arc.}\, qr = \frac{mk}{aa}(qr - fg),$$

erit

$$2\,\text{Arc.}\, fg - \text{Arc.}\, pqr = \frac{mk}{aa}(pq + qr - 2fg).$$

Hinc pari modo definiamus punctum s, ut sit abscissa

$$AS = s = \frac{rK + kR}{a^4 - mkkrr} = \frac{a^4(rr-kk)}{rK - kR}$$

et

$$S = \frac{a^4(ss-rr) - kk(a^4 - mrrss)}{2kr} = \frac{rs(ss-rr)K - ks(ss-kk)R}{kr(rr-kk)},$$

et quia erit

$$\text{Arc.}\, fg - \text{Arc.}\, rs = \frac{mk}{aa}(rs - fg),$$

habebitur

$$3\,\text{Arc.}\, fg - \text{Arc.}\, pqrs = \frac{mk}{aa}(pq + qr + rs - 3fg).$$

Q. E. I.

COROLLARIUM 1

104. Simili modo progrediendo manifestum est definiri a dato posse arcum pt, qui a quadruplo arcus dati fg deficiat quantitate a atque hoc modo operationem continuari posse, quousque lubuerit.

COROLLARIUM 2

105. Si arcus datus fg toti quadranti aequetur, ut sit $f = 0$ ideoque $F = a^4$ et $G = 0$, erit $k = a$ et $K = 0$. Hinc reperitur

$$q = \frac{P}{a(aa-mpp)} = a\sqrt{\frac{aa-pp}{aa-mpp}}$$

$$= \frac{-q(qq-aa)}{p(pp-aa)} P = \frac{-(aa-qq)PP}{ap(aa-mpp)(aa-pp)} = -\frac{a^3(aa-qq)}{p};$$

$$aa-qq = \frac{a(1-m)pp}{aa-mpp}, \quad \text{unde} \quad Q = \frac{-(1-m)a^5 p}{aa-mpp}.$$

$$r = \frac{Q}{a(aa-mqq)} = -p$$

$$R = -aa\sqrt{(aa-pp)(aa-mpp)} = -P.$$

$$\frac{-P}{a(aa-mpp)} = -a\sqrt{\frac{aa-pp}{aa-mpp}} = -q \quad \text{et} \quad S = -Q = \frac{(1-m)a^5 p}{aa-mpp}$$

$$3\,\text{Arc.}\,fg - \text{Arc.}\,pqrs = \frac{m}{a}\,pq = mp\sqrt{\frac{aa-pp}{aa-mpp}}.$$

COROLLARIUM 3

...mctum p quoque ita definiri poterit, ut fiat

$$pq + qr + rs = 3fg,$$

...cus $pqrs$ exacte aequabitur triplo arcus dati fg. Atque ita porro ...i poterit, qui ad arcum datum fg aliam quamvis rationem multi...t.

SCHOLION

...nnia haec problemata, quae hic pro ellipsi tractavi, simili modo ...la resolvi poterunt; ita etiam dato quocunque hyperbolae arcu a ...ovis eiusdem hyperbolae puncto arcus abscindi poterit, qui discrepet ...pso arcu vel ab eius duplo vel triplo vel ab alio quovis multiplo ...eometrice assignabili. Deinde etiam hoc punctum ita assumere ...ifferentia plane in nihilum abeat, quo casu dato quocunque hyper...alius arcus assignari poterit, qui vel eius duplo vel triplo vel ...multiplo exacte sit aequalis. Unde perspicuum est, si proposito ...s sit alius arcus, qui ad illum teneat rationem μ ad 1, similique

modo alius quaeratur arcus, qui ad eundem teneat ration
pacto duos haberi arcus hyperbolicos, qui inter se tenear
sicque infinitis modis bini arcus exhiberi poterunt, qui
cunque numeri ad numerum. Neque vero huiusmodi pro
hyperbola resolvi poterunt, sed omnino pro aliis curvis
ita sint comparata, ut arcus abscissae vel alii cuicunque
bili x respondens contineatur in hac formula

$$\int \frac{dx(\mathfrak{A} + \mathfrak{B}xx + \mathfrak{C}x^4)}{\sqrt{(A + Cxx + Ex^4)}},$$

quae etiam per regulas initio datas ita latius extendi
formam revocetur

$$\int \frac{dx(\mathfrak{A} + \mathfrak{B}xx + \mathfrak{C}x^4 + \mathfrak{D}x^6 + \mathfrak{E}x^8 + etc.)}{\sqrt{(A + Cxx + Ex^4)}};$$

sed in praesentia neque hyperbolae neque aliis huius ge
immorandum esse arbitror.

DEMONSTRATIO THEOREMATIS ET SOLUTIO PROBLEMATIS IN ACTIS ERUD. LIPSIENSIBUS PROPOSITORUM

Commentatio 264 indicis ENESTROEMIANI
Novi commentarii academiae scientiarum Petropolitanae 7 (1758/9), 1761, p. 128—16
Summarium ibidem p. 10—11

SUMMARIUM

Cum in Actis Lips. theorema hoc ac problema sine nomine sint proposita, Cel. A statim se eorum esse inventorem profitetur. Utrumque eximiam ellipseos proprie mplectitur. In theoremate enim docetur, quomodo dimidia ellipsis diametro quac minata ita in duas partes secanda sit, ut partium differentia geometrice assignari ie ipsa divisio cum partium differentia in eo exponitur, ut a geometris demonstratio in retur. Prodiit quidem nuper[2]) in Actis Sociorum Academiae Parisinae huius th tis demonstratio, quae etsi veritatem enunciatam rite ostendat, non tamen ex ge ncipiis hausta videtur. *Unde innumerabilia alia eiusdem generis in ellipsi aliisque* vis invenire licet. Idemque ex eo vel maxime apparet, quod Auctor huius demonstra utionem problematis aggredi non sit ausus, cum tamen ex iisdem principiis nostri is expediri queat. In eo autem quaeritur modus in quadrante elliptico partem geom ignandi, quae exacte semissi quadrantis aequetur. Celeberrimus igitur EULERUS i ipto non solum suo more theorema memoratum demonstrat, sed etiam problema

1) Vide p. 56. A. K.

2) CH. BOSSUT, *Démonstration d'un théoreme de géométrie énoncé dans les actes de Le née 1754*, Mém. prés. par div. sav. Paris. T. 3, 1760, p. 314. A. K.

resolvit idque ope methodi illius novae, quam iam pridem ...
cuius bina nova in hoc volumine specimina edidit, quorum occa
fusius est expositum, quae hic repetere superfluum foret. Adiun
minus notatu digna, veluti id, quod circa finem affert, quo in e
sit totius perimetri ellipticae pars tertia.

Theorema istud et problema versantur circa arcus
ellipseos quaeque ita secatur, ut partium differentia sit
hoc vero constructio geometrica arcus postulatur, qui
elliptici. Tam demonstratio theorematis quam solutio
ex iis, quae iam aliquoties[1]) de comparatione linearum
quoniam methodus, qua hoc argumentum pertractavi,
etiam plurimum recondita videbatur, has propositio
stitueram, ut alii quoque vires suas in iis evolvend
methodis, quibus forte eo pertingerent, fines Analyse
autem nemo adhuc sit inventus, qui hoc negotium c
etiamsi vix dubitare liceat, quin plures id frustra te
quidem inde concludere videor praeter methodum, q
ullam aliam viam ad huiusmodi speculationes patere.
dus perquam indirecte et quasi per ambages proced
eam cuiquam, qui huiusmodi problemata sit aggressu
venire, mirum non est has quaestiones ab aliis inta
igitur iam aliquot specimina huius methodi singularis
pretium fore arbitror, si eius explicationem magis ill
dationem problematis ac theorematis propositi acc
ut ea saepius tractando magis trita et familiaris re
ope ad maxime absconditas proprietates ellipsis alia
inopinato sim deductus, nullum est dubium, quin in
dissimae indaginis contineantur, quae non nisi post fre
inde eruere liceat.

---

1) L. Euleri Commentationes 252, 263, 261 (indicis
108, 153.

LEMMA 1

binae variabiles x et y ita a se invicem pendeant, ut sit

$$0 = \alpha + \beta(xx+yy) + 2\gamma xy + \delta xxyy,$$

mma sive differentia harum formularum integralium

$$\frac{dy}{\ldots\alpha\beta+(\gamma\gamma-\alpha\delta-\beta\beta)yy-\beta\delta y^4)} \pm \int \frac{dx}{\sqrt{(-\alpha\beta+(\gamma\gamma-\alpha\delta-\beta\beta)xx-\beta\delta x^4)}}$$

antitati constanti.

DEMONSTRATIO

enim sit,

$$0 = \alpha + \beta(xx+yy) + 2\gamma xy + \delta xxyy,$$

atramque radicem extrahendo

$$y = \frac{-\gamma x \pm \sqrt{(-\alpha\beta+(\gamma\gamma-\alpha\delta-\beta\beta)xx-\beta\delta x^4)}}{\beta+\delta xx},$$

$$x = \frac{-\gamma y \pm \sqrt{(-\alpha\beta+(\gamma\gamma-\alpha\delta-\beta\beta)yy-\beta\delta y^4)}}{\beta+\delta yy},$$

itur fore

$$y + \gamma x + \delta xxy = \pm\sqrt{\left(-\alpha\beta+(\gamma\gamma-\alpha\delta-\beta\beta)xx-\beta\delta x^4\right)},$$

$$x + \gamma y + \delta xyy = \pm\sqrt{\left(-\alpha\beta+(\gamma\gamma-\alpha\delta-\beta\beta)yy-\beta\delta y^4\right)}.$$

ro aequatio proposita differentietur, oritur

$$0 = \beta x dx + \beta y dy + \gamma y dx + \gamma x dy + \delta xyy dx + \delta xxy dy$$

$$0 = dx(\beta x + \gamma y + \delta xyy) + dy(\beta y + \gamma x + \delta xxy),$$

in hanc

$$\frac{dy}{\beta x+\gamma y+\delta xyy} + \frac{dx}{\beta y+\gamma x+\delta xxy} = 0.$$

tur loco denominatorum formulae illae irrationales, ut prodeant duo ifferentialia, in quibus variabiles x et y sint a se invicem separatae, lis integralibus obtinebitur

$$\frac{dy}{\ldots(\gamma\gamma-\alpha\delta-\beta\beta)yy-\beta\delta y^4)} \pm \int \frac{dx}{\sqrt{(-\alpha\beta+(\gamma\gamma-\alpha\delta-\beta\beta)xx-\beta\delta x^4)}} = \text{Const.}$$

2. Summa harum formularum integralium erit constans, si in radicis extractione signis radicalibus paria tribuantur signa; sin aut statuantur disparia, tum differentia formularum integralium erit cons

COROLLARIUM 2

3. Si ponamus

$$-\alpha\beta = Ak, \quad \gamma\gamma - \alpha\delta - \beta\beta = Bk, \quad -\beta\delta = Ck,$$

inde fiet

$$\alpha = \frac{-Ak}{\beta}, \quad \delta = \frac{-Ck}{\beta} \quad \text{et} \quad \gamma = \frac{\sqrt{(ACkk + Bk\beta\beta + \beta^4)}}{\beta}.$$

Quare si relatio inter x et y hac aequatione exprimatur

$$0 = -Ak + \beta\beta(xx + yy) + 2xy\sqrt{(ACkk + Bk\beta\beta + \beta^4)} - Ckxx$$

erit

$$\int \frac{dy}{\sqrt{(A + Byy + Cy^4)}} + \int \frac{dx}{\sqrt{(A + Bxx + Cx^4)}} = \text{Const.}$$

COROLLARIUM 3

4. Substitutis autem loco α, δ, γ his valoribus erit

$$y = \frac{-x\sqrt{(ACkk + Bk\beta\beta + \beta^4)} \pm \beta\sqrt{k(A + Bxx + Cx^4)}}{\beta\beta - Ckxx},$$

$$x = \frac{-y\sqrt{(ACkk + Bk\beta\beta + \beta^4)} \pm \beta\sqrt{k(A + Byy + Cy^4)}}{\beta\beta - Ckyy},$$

qui ergo sunt valores illi aequationi integrali convenientes, et qu formulis inest constans arbitraria $\frac{\beta\beta}{k}$, eae integrale completum exhil censendae.

COROLLARIUM 4

5. Ad has formulas commodiores reddendas, quia posito x $y = \pm\frac{\sqrt{Ak}}{\beta}$, ponatur $\frac{\sqrt{Ak}}{\beta} = f$ et prodibit

$$y = \frac{x\sqrt{A(A + Bff + Cf^4)} \pm f\sqrt{A(A + Bxx + Cx^4)}}{A - Cffxx},$$

$$x = \frac{y\sqrt{A(A + Bff + Cf^4)} \pm f\sqrt{A(A + Byy + Cy^4)}}{A - Cffyy},$$

ces huius aequationis

$$Aff + A(xx+yy) - 2xy\sqrt{A(A+Bff+Cf^4)} - Cffxxyy.$$

COROLLARIUM 5

o relatio inter x et y hac aequatione exprimatur

$$Aff + A(xx+yy) + 2xy\sqrt{A(A+Bff+Cf^4)} - Cffxxyy,$$

$$\int \frac{dy}{\sqrt{(A+Byy+Cy^4)}} + \int \frac{dx}{\sqrt{(A+Bxx+Cx^4)}} = \text{Const.}$$

$$\frac{dy}{\sqrt{(A+Byy+Cy^4)}} + \frac{dx}{\sqrt{(A+Bxx+Cx^4)}} = 0.$$

COROLLARIUM 6

m ergo si habeatur haec aequatio differentialis

$$\frac{dy}{\sqrt{(A+Byy+Cy^4)}} + \frac{dx}{\sqrt{(A+Bxx+Cx^4)}} = 0,$$

et y ita se habebit, ut sit

$$y = \frac{-x\sqrt{A(A+Bff+Cf^4)} + f\sqrt{A(A+Bxx+Cx^4)}}{A-Cffxx}$$

$$x = \frac{-y\sqrt{A(A+Bff+Cf^4)} + f\sqrt{A(A+Byy+Cy^4)}}{A-Cffyy}.$$

COROLLARIUM 7

proposita hac aequatione differentiali

$$\frac{dy}{\sqrt{(A+Byy+Cy^4)}} - \frac{dx}{\sqrt{(A+Bxx+Cx^4)}} = 0$$

ralis completa erit

$$y = \frac{x\sqrt{A(A+Bff+Cf^4)} + f\sqrt{A(A+Bxx+Cx^4)}}{A-Cffxx}$$

$$x = \frac{y\sqrt{A(A+Bff+Cf^4)} - f\sqrt{A(A+Byy+Cy^4)}}{A-Cffyy}.$$

9. Retinebo determinationes huius postremi casus, quibus efficitu relatio inter binas variabiles x et y fuerit

$$0 = -Aff + A(xx+yy) - 2xy\sqrt{A(A+Bff+Cf^4)} - Cffxxy$$

sive

$$y = \frac{x\sqrt{A(A+Bff+Cf^4)} + f\sqrt{A(A+Bxx+Cx^4)}}{A-Cffxx}$$

et

$$x = \frac{y\sqrt{A(A+Bff+Cf^4)} - f\sqrt{A(A+Byy+Cy^4)}}{A-Cffyy},$$

tum hanc aequationem differentialem locum habere

$$\frac{dy}{\sqrt{(A+Byy+Cy^4)}} - \frac{dx}{\sqrt{(A+Bxx+Cx^4)}} = 0$$

seu sumtis integralibus fore

$$\int\frac{dy}{\sqrt{(A+Byy+Cy^4)}} - \int\frac{dx}{\sqrt{(A+Bxx+Cx^4)}} = \text{Const.}$$

Pro hoc ergo casu erit

$$\sqrt{(A+Bxx+Cx^4)} = \frac{y(A-Cffxx) - x\sqrt{A(A+Bff+Cf^4)}}{f\sqrt{A}}$$

et

$$\sqrt{(A+Byy+Cy^4)} = \frac{-x(A-Cffyy) + y\sqrt{A(A+Bff+Cf^4)}}{f\sqrt{A}}$$

sicque fiet

$$\frac{f\,dy\sqrt{A}}{y\sqrt{A(A+Bff+Cf^4)} - x(A-Cffyy)} + \frac{f\,dx\sqrt{A}}{x\sqrt{A(A+Bff+Cf^4)} - y(A-Cffx}$$

LEMMA 2

10. *Eadem manente relatione inter binas variabilis x et y, ut sit*

$$0 = -Aff + A(xx+yy) - 2xy\sqrt{A(A+Bff+Cf^4)} - Cffxx$$

seu

$$y = \frac{x\sqrt{A(A+Bff+Cf^4)} + f\sqrt{A(A+Bxx+Cx^4)}}{A-Cffxx}$$

et

$$x = \frac{y\sqrt{A(A+Bff+Cf^4)} - f\sqrt{A(A+Byy+Cy^4)}}{A-Cffyy},$$

harum formularum integralium

$$\int \frac{dy(\mathfrak{A}+\mathfrak{B}yy)}{\sqrt{(A+Byy+Cy^4)}} - \int \frac{dx(\mathfrak{A}+\mathfrak{B}xx)}{\sqrt{(A+Bxx+Cx^4)}}$$

nabilis.

DEMONSTRATIO

ostendendum ponamus hanc differentiam $= V$, ut sit

$$\frac{dy(\mathfrak{A}+\mathfrak{B}yy)}{\sqrt{(A+Byy+Cy^4)}} - \frac{dx(\mathfrak{A}+\mathfrak{B}xx)}{\sqrt{(A+Bxx+Cx^4)}} = dV.$$

,

$$\frac{dy}{\sqrt{(A+Byy+Cy^4)}} = \frac{dx}{\sqrt{(A+Bxx+Cx^4)}},$$

$$= \frac{\mathfrak{B}(yy-xx)dx}{\sqrt{(A+Bxx+Cx^4)}} = \frac{\mathfrak{B}f(yy-xx)dx\sqrt{A}}{y(A-Cffxx)-x\sqrt{A(A+Bff+Cf^4)}}.$$

$xy = u$, ut sit $y = \frac{u}{x}$ et

$$- Aff + Axx + \frac{Auu}{xx} - 2u\sqrt{A(A+Bff+Cf^4)} - Cffuu,$$

ne differentiata fit

$$dx - \frac{Auudx}{x^3} + \frac{Audu}{xx} - du\sqrt{A(A+Bff+Cf^4)} - Cffudu;$$

y per x multiplicando oritur

$$\frac{dx}{y(A-Cffxx)-x\sqrt{A(A+Bff+Cf^4)}} = \frac{du}{A(yy-xx)},$$

cata per $\mathfrak{B}f(yy-xx)\sqrt{A}$ praebet

$$dV = \frac{\mathfrak{B}fdu}{\sqrt{A}} \quad \text{et} \quad V = \text{Const.} + \frac{\mathfrak{B}fxy}{\sqrt{A}}.$$

a pro formularum integralium differentia habebimus

$$\int \frac{dy(\mathfrak{A}+\mathfrak{B}yy)}{\sqrt{(A+Byy+Cy^4)}} - \int \frac{dx(\mathfrak{A}+\mathfrak{B}xx)}{\sqrt{(A+Bxx+Cx^4)}} = \text{Const.} + \frac{\mathfrak{B}fxy}{\sqrt{A}},$$

est geometrice assignabilis.

11. Propositis ergo duabus formulis integralibus

$$\int \frac{dy(\mathfrak{A}+\mathfrak{B}yy)}{\sqrt{(A+Byy+Cy^4)}} \quad \text{et} \quad \int \frac{dx(\mathfrak{A}+}{\sqrt{(A+Bx}}$$

eiusmodi relatio inter x et y exhiberi potest, ut
rentia fiat geometrice assignabilis.

COROLLARIUM 2

12. Hunc scilicet in finem talis relatio inter
debet, ut sit

$$0 = -Aff + A(xx+yy) - 2xy\sqrt{A(A+Bff}$$

cuius aequationis resolutio cum sit ambigua, capi d

$$y = \frac{x\sqrt{A(A+Bff+Cf^4)} + f\sqrt{A(A+B}}{A-Cffxx}$$

et

$$x = \frac{y\sqrt{A(A+Bff+Cf^4)} - f\sqrt{A(A+B}}{A-Cffyy}$$

COROLLARIUM 3

13. Quemadmodum hic y per x et f atque x
etiam simili modo f per x et y definiri potest. Er

$$f = \frac{y\sqrt{A(A+Bxx+Cx^4)} - x\sqrt{A(A+B}}{A-Cxxyy}$$

unde patet, si sit $x=0$, fore $y=f$, ex quo casu
ipsius V ingrediens definiri debet.

SCHOLION

14. Simili modo demonstrari potest etiam haru
differentiam

$$\int \frac{dy(\mathfrak{A}+\mathfrak{B}yy+\mathfrak{C}y^4+\mathfrak{D}y^6)}{\sqrt{(A+Byy+Cy^4)}} - \int \frac{dx(\mathfrak{A}+\mathfrak{B}xx+}{\sqrt{(A+B}}$$

assignabilem. Posito enim $xy = u$ erit

$$\frac{fdu}{(yy-xx)\sqrt{A}}(\mathfrak{B}(yy-xx) + \mathfrak{C}(y^4-x^4) + \mathfrak{D}(y^6-x^6))$$

$$V = \frac{fdu}{\sqrt{A}}(\mathfrak{B} + \mathfrak{C}(yy+xx) + \mathfrak{D}(y^4+xxyy+x^4)).$$

o canonica habemus

$$xx + yy = \frac{Aff + 2u\sqrt{A(A+Bff+Cf^4)} + Cffuu}{A}.$$

tis gratia $\sqrt{A(A+Bff+Cf^4)} = Fff$, ut sit

$$xx + yy = \frac{ff}{A}(A + 2Fu + Cuu),$$

$$xxyy + x^4 = (xx+yy)^2 - uu$$

$$dV = \frac{fdu}{\sqrt{A}}\left\{\begin{array}{l} \mathfrak{B} + \frac{\mathfrak{C}ff}{A}(A+2Fu+Cuu) \\ + \frac{\mathfrak{D}f^4}{AA}(A+2Fu+Cuu)^2 - \mathfrak{D}uu \end{array}\right\}$$

do

$$\left.\begin{array}{l} \mathfrak{B}u + \frac{\mathfrak{C}ff}{A}(Au + Fuu + \frac{1}{3}Cu^3) - \frac{1}{3}\mathfrak{D}u^3 \\ \frac{f^4}{A}(AAu + 2AFuu + \frac{2}{3}(AC+2FF)u^3 + CFu^4 + \frac{1}{5}CCu^5) \end{array}\right\}$$

senti instituto, quo ellipsis nobis est proposita, formulae in o sufficiunt.

LEMMA 3

ig. 1) *sit centrum ellipseos* $CA = a$, $CB = b$ *atque ad ur tangens* AD, *in qua definita* $AZ = z$, *et ex* Z *ularis erigatur* ZMV, *erit c* $AZ = z$ *respondens*

$$\frac{z}{b}\sqrt{\frac{b^4 - (bb-aa)zz}{bb-zz}}.$$

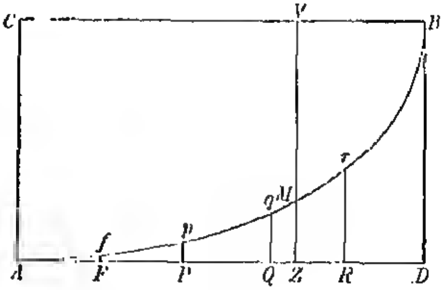


Fig. 1.

DEMONSTRATIO

Ponatur $ZM = v$ et ipse arcus $AM = s$; erit ex natura e

$$VM = a - v = \frac{a}{b}\sqrt{(bb - zz)}$$

hincque

$$v = a - \frac{a}{b}\sqrt{(bb - zz)} \quad \text{et} \quad dv = \frac{azdz}{b\sqrt{(bb - zz)}}.$$

Quare cum sit $ds = \sqrt{(dz^2 + dv^2)}$, erit

$$ds = dz\sqrt{\left(1 + \frac{aazz}{bb(bb - zz)}\right)} = \frac{dz}{b}\sqrt{\frac{b^4 - (bb - aa)zz}{bb - zz}}$$

et integrando

$$s = \text{Arc. } AM = \int \frac{dz}{b}\sqrt{\frac{b^4 - (bb - aa)zz}{bb - zz}}$$

integrali ita accepto, ut evanescat posito $z = 0$.

COROLLARIUM 1

16. Ad hanc formulam contrahendam ponamus hic et perpetuo $\frac{bb - aa}{bb} = n$, ut sit $a = b\sqrt{(1 - n)}$, eritque arcus abs respondens

$$AM = \int dz\sqrt{\frac{bb - nzz}{bb - zz}}.$$

Seu cum sit

$$AM = \int \frac{dz(bb - nzz)}{\sqrt{(b^4 - (n+1)bbzz + nz^4)}},$$

haec expressio ad nostram formam tractatam

$$\int \frac{dz(\mathfrak{A} + \mathfrak{B}zz)}{\sqrt{(A + Bzz + Cz^4)}}$$

reducetur ponendo

$$\mathfrak{A} = bb, \quad \mathfrak{B} = -n, \quad A = b^4, \quad B = -(n + 1)bb, \quad C$$

ita ut sit

$$\sqrt{(A + Bzz + Cz^4)} = \sqrt{(bb - zz)(bb - nzz)}.$$

. Cum ob $a = b\sqrt{(1-n)}$ sit

$$dv = \frac{zdz\sqrt{(1-n)}}{\sqrt{(bb-zz)}} \quad \text{et} \quad ds = dz\sqrt{\frac{bb-nzz}{bb-zz}},$$

nguli AMZ sinus $= \frac{dz}{ds} = \sqrt{\frac{bb-zz}{bb-nzz}}$, cosinus $= \frac{dv}{ds} = \frac{z\sqrt{(1-n)}}{\sqrt{(bb-nzz)}}$ et s $= \frac{dz}{dv} = \frac{\sqrt{(bb-zz)}}{z\sqrt{(1-n)}}$, quas formulas probe notasse iuvabit:

$$\text{sinus} \quad AMZ = \sqrt{\frac{bb-zz}{bb-nzz}},$$

$$\text{cosinus} \quad AMZ = \frac{z\sqrt{(1-n)}}{\sqrt{(bb-nzz)}},$$

$$\text{tangens} \quad AMZ = \frac{\sqrt{(bb-zz)}}{z\sqrt{(1-n)}}.$$

COROLLARIUM 3

8. Designabo porro arcum AM, qui abscissae cuique $AZ = z$ respondet xpressione $\Pi : z$, ut sit

$$AM = \Pi : z = \int dz\sqrt{\frac{bb-nzz}{bb-zz}}.$$

i variae abscissae ponantur

$$AF = f, \quad AP = p, \quad AQ = q, \quad AR = r, \quad AD = CB = b,$$

arcus respondentes

$$Af = \Pi : f, \quad Ap = \Pi : p, \quad Aq = \Pi : q, \quad Ar = \Pi : r, \quad AMB = \Pi : b.$$

COROLLARIUM 4

9. Hoc modo etiam arcus, qui non in puncto A terminantur, commod ni poterunt; sic enim erit

$$\text{arcus } fp = \Pi : p - \Pi : f, \quad \text{arcus } pq = \Pi : q - \Pi : p,$$

$$\text{arcus } qr = \Pi : r - \Pi : q, \quad \text{arcus } pr = \Pi : r - \Pi : p,$$

$$\text{arcus } Bp = \Pi : b - \Pi : p, \quad \text{arcus } Bq = \Pi : b - \Pi : q.$$

at enim $\Pi : b$ arcum totius quadrantis AMB ideoque $4\Pi : b$ totam s peripheriam.

20. *Proposito in ellipsi arcu Af (Fig. 1, p. 209) in alio quovis puncto p arcum abscindere pq, qui ab illo arcu geometrice assignabili.*

SOLUTIO

Positis abscissis, quae punctis f, p et q respond et $AQ = q$ ex datis f et p convenienter determinari pro lemmate secundo sit

$$\mathfrak{A} = bb, \quad \mathfrak{B} = -n, \quad A = b^4, \quad B = -(n+1)b$$

capiatur q ita, ut sit

$$q = \frac{bbp\sqrt{(bb-ff)(bb-nff)} + bbf\sqrt{(bb-pp)(}}{b^4 - nffpp}$$

eritque per lemmatis conclusionem

$$\int dq\sqrt{\frac{bb-nqq}{bb-qq}} - \int dp\sqrt{\frac{bb-npp}{bb-pp}} = \text{Const.}$$

At est

$$\int dq\sqrt{\frac{bb-nqq}{bb-qq}} = \varPi : q \quad \text{et} \quad \int dp\sqrt{\frac{bb-n}{bb-p}}$$

unde

$$\varPi : q - \varPi : p = \text{Const.} - \frac{nfpq}{bb},$$

ubi tantum superest, ut constans debite definiatur. V fit $q = f$, ad quem casum aequatione translata fiet $\varPi$ introducto habebimus

$$\varPi : q - \varPi : p = \varPi : f - \frac{nfpq}{bb}$$

sive

$$\text{Arc. } pq = \text{Arc. } Af - \frac{nfpq}{bb}.$$

COROLLARIUM 1

21. Quia vero eidem abscissae $AQ = q$ bina in ell ad hoc punctum perfecte determinandum etiam appli finiri debet. Est vero

$$Qq = a - \frac{a}{b} \sqrt{(bb - qq)} = \left(b - \sqrt{(bb - qq)}\right) \sqrt{(1 - n)}$$

t

$$\sqrt{(bb - qq)} = \frac{b^3 \sqrt{(bb - ff)(bb - pp)} - bfp \sqrt{(bb - nff)(bb - npp)}}{b^4 - nffpp}.$$

'um etiam notari meretur

$$\sqrt{(bb - nqq)} = \frac{b^3 \sqrt{(bb - nff)(bb - npp)} - nbfp \sqrt{(bb - ff)(bb - pp)}}{b^4 - nffpp};$$

i igitur valor ipsius $\sqrt{(bb - qq)}$ fit negativus, punctum q in superiori e
uadrante capi debet.

COROLLARIUM 2

22. Hic igitur primo relatio notari debet, quae inter tria punct
t q intercedit, quae ita est comparata, ut ex binis datis tertium inveniri

I. Si f et p sint data, erit

$$q = \frac{bbp \sqrt{(bb - ff)(bb - nff)} + bbf \sqrt{(bb - pp)(bb - npp)}}{b^4 - nffpp},$$

$$\sqrt{(bb - qq)} = \frac{b^3 \sqrt{(bb - ff)(bb - pp)} - bfp \sqrt{(bb - nff)(bb - npp)}}{b^4 - nffpp},$$

$$\sqrt{(bb - nqq)} = \frac{b^3 \sqrt{(bb - nff)(bb - npp)} - nbfp \sqrt{(bb - ff)(bb - pp)}}{b^4 - nffpp}.$$

II. Si f et q sint data, erit

$$p = \frac{bbq \sqrt{(bb - ff)(bb - nff)} - bbf \sqrt{(bb - qq)(bb - nqq)}}{b^4 - nffqq},$$

$$\sqrt{(bb - pp)} = \frac{b^3 \sqrt{(bb - ff)(bb - qq)} + bfq \sqrt{(bb - nff)(bb - nqq)}}{b^4 - nffqq},$$

$$\sqrt{(bb - npp)} = \frac{b^3 \sqrt{(bb - nff)(bb - nqq)} + nbfq \sqrt{(bb - ff)(bb - qq)}}{b^4 - nffqq}.$$

III. Si p et q sint data, erit

$$f = \frac{bbq \sqrt{(bb - pp)(bb - npp)} - bbp \sqrt{(bb - qq)(bb - nqq)}}{b^4 - nppqq},$$

$$\sqrt{(bb - ff)} = \frac{b^3 \sqrt{(bb - pp)(bb - qq)} + bpq \sqrt{(bb - npp)(bb - nqq)}}{b^4 - nppqq},$$

$$\sqrt{(bb - nff)} = \frac{b^3 \sqrt{(bb - npp)(bb - nqq)} + nbpq \sqrt{(bb - pp)(bb - qq)}}{b^4 - nppqq}.$$

Hae autem formulae omnes ex hac nascuntur

$$0 = -b^4ff + b^4pp + b^4qq - 2bbpq\sqrt{(bb-ff)(bb-nff)} -$$

quae adeo ad hanc rationalem, in qua f, p et q aequaliter in

$$0 = b^8(f^4 + p^4 + q^4) + 4(n+1)b^6ffppqq - 2b^8(ffpp + ffqq$$
$$- 2nb^4ffppqq(ff + pp + qq) + nnf^4p^4q^4.$$

COROLLARIUM 3

23. Harum formularum igitur ope, si trium punctorum sint bina quaecunque, tertium inveniri poterit, ut arcuum Af geometrice fiat assignabilis. Erit enim

$$\text{Arc.}\, Af - \text{Arc.}\, pq = \text{Arc.}\, Ap - \text{Arc.}\, fq = \frac{nfpq}{bb}.$$

COROLLARIUM 4

24. Denotat autem b semiaxem ellipsis CB et posito fecimus $\frac{bb-aa}{bb} = n$; unde, si $n = 0$, ellipsis abit in circulum et torum differentia evanescit. Ellipsis autem abibit in parabola parameter $= c$, si $bb = ac$ et $a = \infty$. Hoc ergo casu fiet

$$n = \frac{c-a}{c} = -\frac{a}{c} \quad \text{et} \quad \frac{n}{bb} = -\frac{1}{cc},$$

ideoque

$$n = -\frac{bb}{cc} \quad \text{et} \quad \sqrt{(bb-ff)} = b, \quad \sqrt{(bb-nff)} = b\sqrt{\Big(1}$$

unde formulae superiores ad parabolam transferri poterunt.

COROLLARIUM 5

25. Si easdem formulas ad hyperbolam accommodare veli b ita imaginarium statui oportet, ut eius quadratum bb fiat tiva. Seu, quod eodem redit, in nostris formulis ubique lo $-bb$ et semiaxis a capiatur negative; tum vero n erit n maior.

PROBLEMA 2

In quadrante elliptico AB (Fig. 2) *dato puncto quocunque f invenire aliu* *g, ut arcuum Af et Bg differentia sit geometrice assignabilis.*

SOLUTIO

praecedente problemate hoc facile resolvitur; positis enim semiaxibu $CB = b$ et $\frac{bb-aa}{bb} = n$ punctum q in praecedente problemate in B omoveri oportet, ut fiat $q = b$; tum sint super tangente AD vel axe CB sumtae ' et g respondentes $AF = C\mathfrak{F} = f$ et $\mathfrak{G} = g$, ita ut, quod ante erat p, nunc ique ex dato puncto f determinatio per formulas § 22 ita se habebit ob $q = b$

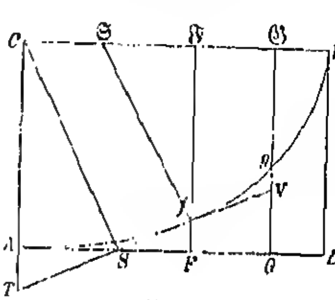


Fig. 2.

$$g = \frac{b^3\sqrt{(bb-ff)(bb-nff)}}{b^4-nbbff} = b\sqrt{\frac{bb-ff}{bb-nff}},$$

$$\sqrt{(bb-gg)} = \frac{bbf\sqrt{(bb-nff)(bb-nbb)}}{b^4-nbbff} = \frac{bf\sqrt{(1-n)}}{\sqrt{(bb-nff)}},$$

$$\sqrt{(bb-ngg)} = \frac{b^3\sqrt{(bb-nff)(bb-nbb)}}{b^4-nbbff} = \frac{bb\sqrt{(1-n)}}{\sqrt{(bb-nff)}}.$$

nguli, quos applicatae Ff et Gg cum curva faciunt, in computum erit

$$g = b \sin. AfF \quad \text{et} \quad f = b \sin. AgG.$$

c sequitur ista constructio pro puncto g inveniendo: Ad punctum f angens fT, donec axi CA producto occurrat in T, tum in ea, s producta capiatur $TV = CB = b$ et per V agatur recta $\mathfrak{G}G$ ax lela eritque punctum g quaesitum, ita ut arcuum Af et Bg diffe- geometrice assignabilis. Verum ex problemate praecedente ob $q = b$ erit haec differentia

$$\text{Arc. } Af - \text{Arc. } Bg = \frac{nfg}{b} = nf\sqrt{\frac{bb-ff}{bb-nff}}.$$

construendam notetur esse

$$Tf = \frac{AF}{\sin. AfF} = f\sqrt{\frac{bb-nff}{bb-ff}}$$

et ex natura ellipsis

$$CT = \frac{ab}{\sqrt{(bb-ff)}} = \frac{bb\sqrt{(1-n)}}{\sqrt{(bb-ff)}}.$$

Hinc si ex centro ellipsis C in tangentem Tf demittatur perpen
ob ang. $CTS =$ ang. AfF eiusque sinum $= \sqrt{\frac{bb-ff}{bb-nff}}$ et cosinum
erit

$$TS = CT \cos. CTS = \frac{bbf(1-n)}{\sqrt{(bb-ff)(bb-nff)}}$$

hincque

$$Sf = Tf - TS = \frac{bbf - nf^3 - bbf + nbbf}{\sqrt{(bb-ff)(bb-nff)}} = \frac{nf(bb-ff)}{\sqrt{(bb-ff)(bb-nff)}} = nf$$

Portio igitur tangentis fS inter perpendiculum CS et punctum
contenta praebebit differentiam arcuum Af et Bg, ita ut sit

$$\text{Arc. } Af - \text{Arc. } Bg = \text{Arc. } Ag - \text{Arc. } Bf = Sf.$$

COROLLARIUM 1

27. Haec differentia arcuum facilius inveniri potest, si in f a
ducatur normalis $f\mathfrak{S}$; tum enim ex natura ellipsis statim consta

$$C\mathfrak{S} = f - \frac{aa}{bb}f = nf.$$

Quare cum CS ipsi $\mathfrak{S}f$ sit parallela et angulus $BCS = CTS =$
ergo sinus $= \sqrt{\frac{bb-ff}{bb-nff}}$, erit

$$Sf = C\mathfrak{S} \sin. BCS = nf\sqrt{\frac{bb-ff}{bb-nff}}.$$

COROLLARIUM 2

28. Simili modo ex puncto g definietur punctum f; si enim
tangens usque ad axem CA atque ab intersectione eius cum axo i
portio alteri semiaxi CB aequalis, haec praecise in recta Ff
ideoque punctum f monstrabit.

COROLLARIUM 3

29. Constructio ergo puncti g ex dato puncto f ita se habebit: unctum f ducatur tangens axi CA producto occurrens in T in eaque scindatur portio TV semiaxi CB aequalis, et recta GS axi CA para r punctum V acta in ellipsi punctum quaesitum g definiet. Tum enim centro ellipsis C in illam tangentem perpendiculum CS demittatur,

$$\text{Arc. } Af - \text{Arc. } Bg = \text{Rectae } Sf$$

nequo etiam

$$\text{Arc. } Af - \text{Recta } fS = \text{Arc. } Bg.$$

COROLLARIUM 4

30. Casus notabilis est, quo bina puncta f et g in unum colliquesc . ut arcus quadrantis AfB (Fig. 3) in puncto f ita secari iubeatur rtium Af et Bf differentia fiat geometrice assignabilis. Hunc in fi natur in solutione $g = f$, unde fit

$$f = b\sqrt{\frac{bb-ff}{bb-nff}}$$

ncque

$$2bbff - nf^4 = b^4 \quad \text{et} \quad \frac{bb}{ff} = 1 + \sqrt{(1-n)} = \frac{a+b}{b}.$$

are pro puncto hoc f capi debet abscissa

$$AF = f = b\sqrt{\frac{b}{a+b}};$$

que ob

$$\sqrt{\frac{bb-ff}{bb-nff}} = \frac{f}{b}$$

t partium differentia

$$Af - Bf = \frac{nff}{b} = \frac{nbb}{a+b},$$

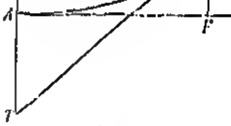


Fig. 3.

ae, cum sit $n = \frac{bb-aa}{bb}$, abit in $Af - Bf = b - a$, ita ut aequalis ev ferentiae semiaxium. Unde puncto f hoc modo definito, ut sit $f = b\sqrt{\frac{\ }{a}}$ t etiam

$$AC + Af = BC + Bf$$

a ducto radio Cf ambo trilinea ACf et BCf pari perimetro includun

31. Quia supra habuimus

$$CT = \frac{ab}{\sqrt{(bb-ff)}},$$

erit pro praesenti casu

$$CT = \sqrt{(aa + ab)} \quad \text{ob} \quad ff = \frac{b^3}{a+b};$$

unde sequens concinna puncti f constructio deducitur: Bisecto in O intervallo $OT = OC + AC$ definiatur in CA producta pun intervallo $Tf = BC$ punctum f in ellipsi designetur eritque f p situm et recta Tf eius tangens.

PROBLEMA 3

32. *Proposita semiellipsi ABa* (Fig. 4) *in eaque sumto quoc definire punctum q ita, ut arcus pBq differat a quadrante elliptic tate geometrice assignabili.*

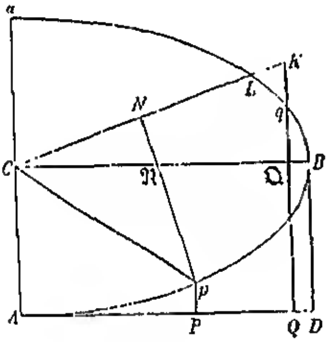


Fig. 4.

SOLUTIO

Positis ut hactenus semiaxibus et ad abbreviandum $n = \frac{bb - aa}{bb}$ in blematis primi promoveatur pun usque eritque vi eius arcuum A rentia geometrice assignabilis, Demissis ergo ad tangentem A perpendiculis pP et qQ sint $AP =$ atque ob $f = b$ habebimus ex § 22

$$q = \frac{b\sqrt{(bb-pp)(bb-npp)}}{bb-npp} = b\sqrt{\frac{bb-pp}{bb-npp}},$$

$$\sqrt{(bb-qq)} = \frac{-p\sqrt{(bb-nbb)(bb-npp)}}{bb-npp} = \frac{-bp\sqrt{(1-n)}}{\sqrt{(bb-npp)}}$$

cuius quantitatis signum — indicat ulteriorem intersectionem p pro puncto q accipi oportere, secus atque in problemate prae igitur $\sqrt{\frac{bb-pp}{bb-npp}}$ exprimat sinum anguli, quem applicata Pp c

it $q = b \sin. ApP$. Ad Qq, si opus est, productam ex centro C diri
cta CK semiaxi $CB = b$ aequalis, ut sit $CK = b$, eritque $\frac{q}{b} = \frac{AQ}{CK} = \sin.$
neque $\sin. CKQ = \sin. ApP$ et $CKQ = ApP$. Ex quo patet rectam
rallelam fore tangenti in puncto p. Quare iuncta Cp eaque ut sem
etro spectata erit CL eius semidiameter coniugata, in qua proinde
cta, si capiatur $CK = CB$, perpendiculum KQ ad CB demissum in c
finiet punctum q. Quo invento ob

$$f = b \quad \text{et} \quad q = b\sqrt{\frac{bb-pp}{bb-npp}}$$

it arcuum differentia

$$\text{Arc. } AB - \text{Arc. } pq = \frac{nfpq}{bb} = np\sqrt{\frac{bb-pp}{bb-npp}} = np \sin. ApP.$$

icatur ad ellipsin in p normalis $p\mathfrak{N}$; erit $C\mathfrak{N} = np$ et producta $p\mathfrak{N}$
g. $C\mathfrak{N}N = $ ang. ApP; quare cum haec pN futura sit normalis in diame
niugatam CL, erit $CN = np \sin. ApP$; unde demisso ex p in CL
ndiculo intervallum CN aequabitur differentiae illorum arcuum, ita u

$$\text{Arc. } AB - \text{Arc. } pq = CN.$$

COROLLARIUM 1

33. Cum igitur punctum p pro lubitu assumi possit, infiniti arcu
hiberi possunt, qui a quadrante AB differunt quantitate geometrice ass
i. Quare etiam hi arcus inter se differunt quantitate geometrice assign

COROLLARIUM 2

34. Ex dato ergo puncto p punctum q ita definitur: Ad ductam
gatur semidiameter coniugata CL in K producenda, ut fiat CK aeq
miaxi CB, ad quem ex K perpendiculum demittatur KQ ellipsin se
q; erit q punctum quaesitum. Atque demisso ex p in CL perpend
r erit $AB - pq = CN$.

COROLLARIUM 3

35. Quoties perpendiculum pN (Fig. 5, p. 220) intra C et K cadit, a
erit minor quadrante AB, contra autem, si ad alteram partem c

coniugata CL, qua producta in K, ut sit $CK = CB$, et ex K ad CL perpendiculo KQ secante ellipsin in q, quia hic perpendiculum a demissum ad alteram partem cadit, erit arcus aq = arcu $AB - Cr$

THEOREMA DEMONSTRANDUM

36. *Si ellipsis $ABaqA$ (Fig. 5) diametro quacunque pa fuerit cunque ducatur diameter coniugata $L\lambda$, cuius semissis CL producatur ut fiat CK alteri semiaxi principali CB aequalis, ad quam ex K perpendiculum KQ ellipsin secans in q, tum ellipsis semiperimeter secabitur in q, ut partium aaq et pBq differentia sit geometrice Ductis enim ex p et a ad diametrum coniugatam $L\lambda$ normalibus pN et ctam Nr illi differentiae ita aequabitur, ut sit*

$$\text{Arc. } aaq - \text{Arc. } pBq = Nr.$$

DEMONSTRATIO

Quia CL est semidiameter coniugata conveniens semidiametro constructione, qua punctum q est definitum, patet per § 34 fore

$$\text{Arc. } AB - \text{Arc. } pq = C$$

Fig. 5.

Deinde, quia CL est quoque semidiameter coniugata conveniens semidiametro § 35 patet esse

$$\text{Arc. } aq - \text{Arc. } AB = C$$

Addantur hae duae aequationes tabit.

$$\text{Arc. } aq - \text{Arc. } pq = CN + Cr = Nr.$$

COROLLARIUM

37. Perinde est, utri semiaxi principali semidiameter CL eiusve portio aequalis capiatur, dummodo ex eius termino ad

perpendiculum demittatur. Ita in CL potuisset abscindi porti axi minori $C\alpha$ aequalis; recta enim qkg per k ad $C\alpha$ normaliter llipsi idem punctum g prodidisset.

SCHOLION

38. En ergo demonstrationem completam theorematis in Actis . propositi, quae ita est comparata, ut nullo modo ex vulgaribus e rietatibus derivari potuisset, neque etiam Analysis infinitorum m ii attulerit, nisi hoc ipso modo, quo hic sum usus, in subsidium vo profundis quidem speculationibus Ill. Comitis FAGNANI hanc quoqu trationem deducere liceret; verum inde vix via pateret ad pro m propositum resolvendum, in cuius ergo gratiam sequentia sunt nda.

PROBLEMA 4

39. *Arcum ellipticum quemcunque Ag (Fig. 6) ad alterum axem princ terminatum ita secare in f, ut partium Af et fg differentia sit geor nabilis.*

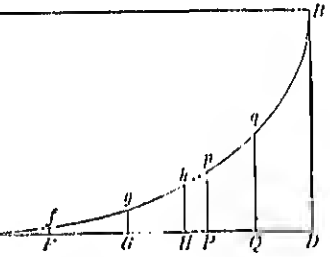


Fig. 6.

SOLUTIO

Positis semiaxibus $CA = a$, $CB =$ brevitatis gratia $n = \frac{bb-aa}{bb}$ in verticis gente AD sumantur abscissae ac ponat scissa toti arcui Ag dato respondens A quaesita autem, quae puncto f responde $AF = f$. Cum igitur differentia arcuum fg debeat esse geometrice assignabilis, qu notur in probl. 1 sumendo ibi $p = f$ et ponendo $q = g$, unde obtine formulas

$$g = \frac{2bbf\sqrt{(bb-ff)(bb-nff)}}{b^4 - nf^4},$$

$$\sqrt{(bb-gg)} = \frac{b^3(bb-ff) - bff(bb-nff)}{b^4 - nf^4} = \frac{b(b^4 - 2bbff + nf^4)}{b^4 - nf^4},$$

$$\sqrt{(bb-ngg)} = \frac{b^3(bb-nff) - nbff(bb-ff)}{b^4 - nf^4} = \frac{b(b^4 - 2nbbff + nf^4)}{b^4 - nf^4}.$$

Ex quibus combinatione oritur

$$\sqrt{(bb-ngg)} - n\sqrt{(bb-gg)} = \frac{(1-n)b(b^4+nf^4)}{b^4-nf^4}$$

hincque

$$\frac{nf^4}{b^4} = \frac{\sqrt{(bb-ngg)} - n\sqrt{(bb-gg)} - (1-n)b}{\sqrt{(bb-ngg)} - n\sqrt{(bb-gg)} + (1-n)b},$$

quae formula reducitur ad

$$\frac{nnf^4}{b^4} = \frac{(\sqrt{(bb-ngg)} - n\sqrt{(bb-gg)} - (1-n)b)^2}{2bb-(1+n)gg-2\sqrt{(bb-gg)(bb-ngg)}},$$

unde radice quadrata extracta fit

$$\frac{nff}{bb} = \frac{\sqrt{(bb-ngg)} - n\sqrt{(bb-gg)} - (1-n)b}{\sqrt{(bb-ngg)} - \sqrt{(bb-gg)}} = \frac{(b-\sqrt{(bb-gg)})(b-}{gg}$$

ex qua porro elicimus

$$\frac{bb-nff}{bb} = \frac{(1-n)(b-\sqrt{(bb-gg)})}{\sqrt{(bb-ngg)}-\sqrt{(bb-gg)}} = \frac{(b-\sqrt{(bb-gg)})(\sqrt{(bb-ng}}{gg}$$

$$\frac{n(bb-ff)}{bb} = \frac{(1-n)(b-\sqrt{(bb-ngg)})}{\sqrt{(bb-ngg)}-\sqrt{(bb-gg)}} = \frac{(b-\sqrt{(bb-ngg)})(\sqrt{(bb-n}}{gg}$$

Punctum igitur quaesitum f ita determinabitur, ut sit

$$f = \frac{b}{g\sqrt{n}}\sqrt{\left(b-\sqrt{(bb-gg)}\right)\left(b-\sqrt{(bb-ngg)}\right)},$$

$$\sqrt{(bb-ff)} = \frac{b}{g\sqrt{n}}\sqrt{\left(b-\sqrt{(bb-ngg)}\right)\left(\sqrt{(bb-gg)}+\sqrt{(}\right.}$$

$$\sqrt{(bb-nff)} = \frac{b}{g}\sqrt{\left(b-\sqrt{(bb-gg)}\right)\left(\sqrt{(bb-gg)}+\sqrt{(bb}\right.}$$

icto f ita determinato ob $p=f$ et $q=g$ par

$$Af - \text{Arc. } fg = \frac{nffg}{bb} = \frac{(b-\sqrt{(bb-gg)})(b-\sqrt{(bb}}{g}$$

COROLLARIUM 1

40. Casum huius problematis iam solvimus § 30, quo arcus secan
toti quadranti AB assumitur aequalis. Si enim ponamus $g=b$, repe
ut ibi

$$f=b\sqrt{\frac{1-\sqrt{(1-n)}}{n}}=b\sqrt{\frac{b(b-a)}{bb-aa}}=\frac{b\sqrt{b}}{\sqrt{(a+b)}}$$

partium differentia prodit $=b-b\sqrt{(1-n)}=b-a$.

COROLLARIUM 2

41. Si arcus dati Ag alter terminus in superiori quadrante existat e
cui abscissa $AG=g$ respondeat, eaedem hae formulae valent, nisi c
or radicalis $\sqrt{(bb-gg)}$ negative capi debeat radicali $\sqrt{(bb-ngg)}$
stato.

COROLLARIUM 3

42. Ita si proponatur tota semiperipheria, erit $g=0$ et $\sqrt{(bb-gg)}=$
lo pro hoc casu obtinebitur

$$f=\frac{b}{g\sqrt{n}}\sqrt{2b\left(b-\sqrt{(bb-ngg)}\right)}=b,$$

licet arcus Af abibit in quadrantem ellipsis. Sin autem integra ell
ripheria proponeretur, tum esset et $g=0$ et $\sqrt{(bb-gg)}=+b$ sicque
sius f prodiret evanescens, at pro $\sqrt{(bb-ff)}$ capi deberet $-b$.

PROBLEMA 5

43. *Proposito in ellipsi arcu Ag altero termino A in axe principali*
to assignare arcum pq, qui sit praecise semissis arcus dati Ag.

SOLUTIO

Manentibus superioribus denominationibus sint abscissae punctis p
spondentes $AP=p$ et $AQ=q$ atque ex puncto p, quasi esset d
quaeratur q, ut differentia arcuum Af et pq fiat geometrice assignabilis
nim quoque differentia arcuum fg et pq geometrice assignari poteri

quidem secundum problema praecedens arcus datus Ag ita sectus est in f, ut partium Af et fg differentia sit Hunc ergo in finem esse debet

$$q = \frac{bbp\sqrt{(bb-ff)(bb-nff)} + bbf\sqrt{(bb-pp)(}{b^4 - nffpp}$$

seu

$$0 = b^4(pp + qq - ff) - 2bbpq\sqrt{(bb-ff)(bb-n}$$

Quo facto erit

$$\text{Arc. } Af - \text{Arc. } pq = \frac{nfpq}{bb}$$

ideoque

$$2 \text{ Arc. } Af - 2 \text{ Arc. } pq = \frac{2nfpq}{bb}.$$

At ex problemate praecedente habemus

$$\text{Arc. } Af - \text{Arc. } fg = \frac{nffg}{bb},$$

qua aequatione ab illa subtracta relinquitur

$$\text{Arc. } Ag - 2 \text{ Arc. } pq = \frac{2nfpq}{bb} - \frac{nff}{bb}$$

Quae differentia cum in nihilum abire debeat, habebi

$$2nfpq = nffg \quad \text{et} \quad 2pq = fg.$$

Pro pq substituatur iste valor $\frac{1}{2}fg$ et obtinebimus

$$b^4(pp + qq) = b^4ff + bbfg\sqrt{(bb-ff)(bb-nf}$$

existente

$$g = \frac{2bbf\sqrt{(bb-ff)(bb-nff)}}{b^4 - nf^4},$$

vel potius pro f introducatur valor ante inventus

$$f = \frac{b}{g\sqrt{n}}\sqrt{\left(b - \sqrt{(bb-gg)}\right)\left(b - \sqrt{(bb-\right.}}$$

unde fit

$$\sqrt{(bb-ff)(bb-nff)} = \frac{bb\left(\sqrt{(bb-gg)} + \sqrt{(bb-ngg)}\right)}{gg\sqrt{n}}\sqrt{\left(b - \sqrt{(bb\right.}}$$

o ambae abscissae p et q ex hac aequatione duplicata definiri

$$\pm 2pq + qq = \frac{b^4ff \pm b^4fg + bbfg\sqrt{(bb-ff)(bb-nff)} + \frac{1}{4}nf^4gg}{b^4}$$

ista irrationalitate ob

$$bbfg\sqrt{(bb-ff)(bb-nff)} = \frac{1}{2}gg(b^4 - nf^4)$$

$$p + q = \frac{\sqrt{\left(b^4ff + b^4fg + \frac{1}{2}b^4gg - \frac{1}{4}nf^4gg\right)}}{bb},$$

$$q - p = \frac{\sqrt{\left(b^4ff - b^4fg + \frac{1}{2}b^4gg - \frac{1}{4}nf^4gg\right)}}{bb},$$

ne abscissa p et q seorsim facile assignatur.

COROLLARIUM 1

quantitatem subsidiariam f penitus eliminemus, perveniemus ad ormulas

$$pp + qq = \frac{1}{4ngg}\left(b - \sqrt{(bb-gg)}\right)\left(b - \sqrt{(bb-ngg)}\right)$$

$$bb + 3b\sqrt{(bb-gg)} + 3b\sqrt{(bb-ngg)} + \sqrt{(bb-gg)(bb-ngg)}\Big),$$

$$2pq = \frac{b}{\sqrt{n}}\sqrt{\left(b - \sqrt{(bb-gg)}\right)\left(b - \sqrt{(bb-ngg)}\right)}.$$

COROLLARIUM 2

arcus propositus Ag sit semiperipheriae aequalis ideoque

$$= 0 \quad \text{et} \quad \sqrt{(bb-gg)} = -b \quad \text{et} \quad \sqrt{(bb-ngg)} = b - \frac{ngg}{2b},$$

c casu

$$pp + qq = bb \quad \text{et} \quad 2pq = bg = 0$$

$= 0$ et $q = b$. Arcus scilicet pq abibit in quadrantem AB, ut postulat.

PROBLEMA SOLVENDUM

46. *In quadrante elliptico AB* (Fig. 7) *arcum assignare pq,*
semissis arcus quadrantis AB.

SOLUTIO

Ponantur ellipsis semiaxes $CA = a$, $CB = b$ sitque bre
$\frac{bb-aa}{bb} = n$. Tum ad A ducatur tangens in eamque ex punc
et q demissa concipiantur perpe
qQ voceturque $AP = p$ et A
manifestum est hoc problema ess
cedentis, quo punctum g in B
ut hoc sit $g = b$. Quo valore in
§ 44 praebebunt

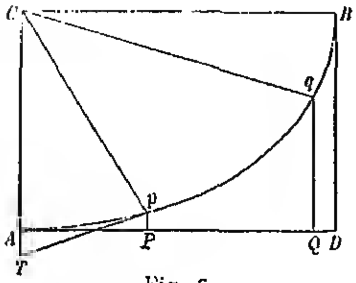


Fig. 7.

$$pp + qq = \frac{1-\sqrt{(1-n)}}{4n}(5bb + \ldots$$

et

$$2pq = bb\sqrt{\frac{1-\sqrt{(1-n \ldots}}{n}}$$

At ob

$$n = \frac{bb-aa}{bb} \quad \text{est} \quad \sqrt{(1-n)} = \frac{a}{b} \quad \text{et} \quad \frac{1-\sqrt{(1-n)}}{n} = b \ldots$$

unde fiet

$$pp + qq = \frac{bb(5b+3a)}{4(a+b)} \quad \text{et} \quad 2pq = \frac{bb\sqrt{b}}{\sqrt{(a+b)}}$$

hincque

$$q + p = \frac{1}{2}b\sqrt{\frac{5b+3a+4\sqrt{b(a+b)}}{a+b}},$$

$$q - p = \frac{1}{2}b\sqrt{\frac{5b+3a-4\sqrt{b(a+b)}}{a+b}}$$

ideoque ipsae abscissae erunt

$$AP = \frac{1}{4}b\sqrt{\frac{5b+3a+4\sqrt{b(a+b)}}{a+b}} - \frac{1}{4}b\sqrt{\frac{5b+3a-4\sqrt{b \ldots}}{a+b}}$$

$$\frac{1}{4}b\sqrt{\frac{5b+3a+4\sqrt{b(a+b)}}{a+b}} + \frac{1}{4}b\sqrt{\frac{5b+3a-4\sqrt{b \ldots}}{a+b}}$$

…metrice per circinum et regulam construi
…o adaequata problematis in Actis Erud.

COROLLARIUM 1

47. Si distantiae binorum punctorum p et q a centro ellipsis desi
notetur posita $AP = p$ fore $Cp = \sqrt{(aa + npp)}$ atque hinc colligitu

$$Cp = \frac{\sqrt{(5aa - 2ab + 5bb + (a-b)\sqrt{(9aa + 14ab + 9bb)})}}{2\sqrt{2}},$$

$$Cq = \frac{\sqrt{(5aa - 2ab + 5bb + (b-a)\sqrt{(9aa + 14ab + 9bb)})}}{2\sqrt{2}}.$$

COROLLARIUM 2

48. Ambae abscissae p et q etiam hoc modo ad constructionem fo
us exprimi possunt, ut sit

$$AP = p = \frac{b\sqrt{(5b + 3a - \sqrt{(9aa + 14ab + 9bb)})}}{2\sqrt{2}(a+b)},$$

$$AQ = q = \frac{b\sqrt{(5b + 3a + \sqrt{(9aa + 14ab + 9bb)})}}{2\sqrt{2}(a+b)}.$$

COROLLARIUM 3

49. Si ad puncta p et q tangentes ducantur ad occursum axis
gnitudo harum tangentium commode exprimitur. Reperietur enim

$$Tp = \frac{\sqrt{(9aa + 14ab + 9bb)} - 3a - b}{4},$$

puncto autem q erit eadem tangens

$$= \frac{\sqrt{(9aa + 14ab + 9bb)} + 3a + b}{4}.$$

COROLLARIUM 4

50. Concipiatur tangens Tp (Fig. 8, p. 228) ad alterum usque axe
tinuata et concursus littera Θ notari eritque permutatis literis a et

$$\Theta p = \frac{\sqrt{(9aa + 14ab + 9bb)} + a + 3b}{4}$$

oque $\Theta p - Tp = a + b$.

COROLLARIUM 5

51. Solutio igitur huius problematis ad hanc
tricam reducitur:

In quadrante elliptico AB (Fig. 8) *duo eiusmodi*
ut ad ea ductis tangentibus $Tp\Theta$, $tq\theta$, *quoad axibus*
utroque

Θp -

et

tq -

seu ut differe
gentis aequa
principalium.

Hoc pro
p et q simul i
interceptus
AB rationem

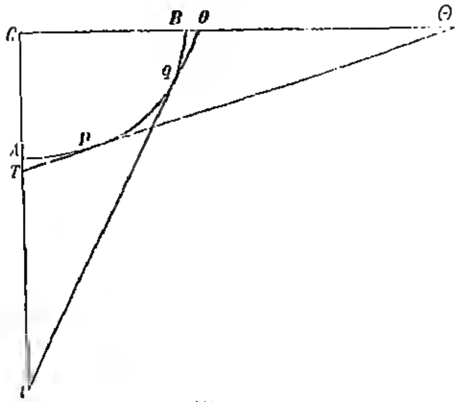


Fig. 8

SCHOLION

52. Demonstrato nunc theoremate solutoque
Erud. Lips. extant proposita, antequam huic inv
problema adhuc multo difficilius pertractabo, quo
iubetur, qui totius perimetri ellipseos sit triens.
arcus assignatur, qui totius perimetri sit semissis
blematis praecedentis etiam octans, haud parum n
quo triens postulatur, cuius solutio, etiamsi ob su
de semissi et quadrante expeditur, non admodum
ad investigationes perquam prolixas et operosas dedu

PROBLEMA 7

53. *Datum ellipsis arcum Ah* (Fig. 6, p. 221)
in A terminatum ita secare in duobus punctis f et
et gh binae quaeris quantitate geometrice assignabili

SOLUTIO

punctis f, g, h ad rectam AD, quae ellipsin in A tangit, demiss
ientis vocentur abscissae $AF = f$, $AG = g$ et $AH = h$, quarum ha
datur, illas vero duas f et g determinari oportet. Cum autem arcu
g differentia geometrica esse debeat, erit ex praecedentibus

$$g = \frac{2bbf\sqrt{(bb-ff)(bb-nff)}}{b^4 - nf^4}$$

$$Af - fg = \frac{nffg}{bb}.$$

quia arcuum Af et gh differentia debet esse geometrica, erit per fo
uperiores

$$g = \frac{bbh\sqrt{(bb-ff)(bb-nff)} - bbf\sqrt{(bb-hh)(bb-nhh)}}{b^4 - nffhh}$$

$$Af - gh = \frac{nfgh}{bb}.$$

itur quoque tertia differentia erit

$$fg - gh = \frac{nfg}{bb}(h - f).$$

iam ambo hi valores ipsius g inter se aequentur, obtinebitur aequat
et h, per quam propterea abscissa f determinabitur, qua inven
bscissa g innotescit.

COROLLARIUM 1

Aequatis autem duobus valoribus ipsius g eruetur

$$(b^4h - nf^4h - 2b^4f + 2nf^3hh)\sqrt{(bb-ff)(bb-nff)}$$

$$= (b^4f - nf^5)\sqrt{(bb-hh)(bb-nhh)},$$

untis utrinque quadratis ad duodecimum gradum ascendit.

COROLLARIUM 2

Si sit $h = b$ seu arcus Ah in B terminetur, habebitur ista aequat
nda

$$- nbf^4 - 2b^4f + 2nbbf^3 = 0 \quad \text{seu} \quad nf^4 - 2nbf^3 + 2b^3f - b^4 = 0.$$

56. *In ellipsi arcum pq* (Fig. 9) *assignare, qui sit tertia pars* *metri ellipsis.*

SOLUTIO

Positis semiaxibus $CA = a$, $CB = b$ et brevitatis ergo $n = \frac{bb}{}$ datur primo tota peripheria ellipsis ita in punctis f et g, ut part fag, $g\beta A$ differentiae sint geom signabiles. Statuantur his punc abscissae respondentes $AF = f$ et quatenus haec in plagam opposi Problema igitur praecedens ad h accommodabitur, si ob punctu incidens ponatur $h = 0$ et $\sqrt{(bb -}$ quo facto habebimus

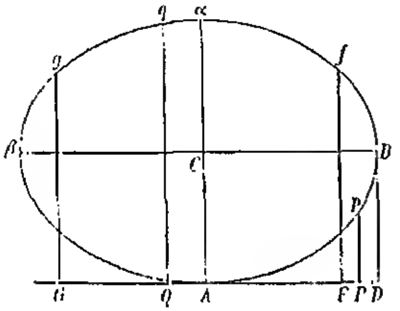


Fig. 9.

$$g = \frac{2bbf\sqrt{(bb-ff)(bb-nff)}}{b^4-nf^4} \quad \text{et}$$

sicque erit $AG = AF = f$ et ternae partes ellipsis ita different, ut

$$fag - ABf = \frac{nf^3}{bb} \quad \text{et} \quad ABf - A\beta g = 0.$$

Cum autem sit $g = -f$, erit

$$2bbf\sqrt{(bb-ff)(bb-nff)} = -(b^4 - nf^4)f,$$

unde quadratis sumtis elicitur

$$nnf^8 - 6nb^4f^4 + 4(n+1)b^6ff - 3b^8 = 0.$$

Ad hanc aequationem resolvendam fingantur eius factores

$$(nf^4 + Pff + Q)(nf^4 - Pff + R) = 0$$

esseque oportet

$$-6nb^4 = n(Q+R) - PP, \quad 4(n+1)b^6 = P(R-Q), \quad -3b^8 =$$

ex quibus fit

$$R + Q = \frac{PP - 6nb^4}{n}, \quad R - Q = \frac{4(n+1)b^6}{P},$$

s ipsarum Q et R in postrema aequatione substituta praebent

$$P^6 - 12nb^4P^4 + 48nnb^8P^2 = 16nn(n+1)^2b^{12},$$

e evenit, ut subtrahendo utrinque $64n^3b^{12}$ cubus relinquatur, cuius cta fiet

$$4nb^4 = 2b^4\sqrt[3]{2nn(1-n)^2} \quad \text{et} \quad P = bb\sqrt{(4n + 2\sqrt[3]{2nn(1-n)^2})}.$$

substituto reperietur

$$R + Q = \frac{2b^4(n - \sqrt[3]{2nn(1-n)^2})}{n},$$

$$R - Q = \frac{2b^4\sqrt{(4nn - 2n\sqrt[3]{2nn(1-n)^2} + \sqrt[3]{4n^4(1-n)^4})}}{n}.$$

ipsa resolutio suppeditat

$$ff = \frac{-P \pm \sqrt{(PP - 4nQ)}}{2n} \quad \text{et} \quad ff = \frac{+P \pm \sqrt{(PP - 4nR)}}{2n},$$

utis valoribus inventis obtinebitur

$$\frac{ff}{} = -\sqrt{(4n + 2\sqrt[3]{2nn(1-n)^2})} \pm \sqrt{(8n - 2\sqrt[3]{2nn(1-n)^2}}$$
$$+ 4\sqrt{(4nn - 2n\sqrt[3]{2nn(1-n)^2} + \sqrt[3]{4n^4(1-n)^4})},$$

$$\frac{ff}{} = +\sqrt{(4n + 2\sqrt[3]{2nn(1-n)^2})} \pm \sqrt{(8n - 2\sqrt[3]{2nn(1-n)^2}}$$
$$- 4\sqrt{(4nn - 2n\sqrt[3]{2nn(1-n)^2} + \sqrt[3]{4n^4(1-n)^4})};$$

m quaternis valoribus alii locum habere nequeunt, nisi qui ff ositivum et minus quam bb.

iam valore idoneo pro f pro punctis quaesitis p et q ponantur $P = p$ et $AQ = q$ ac statuatur

$$b^4(pp + qq - ff) - 2bbpq\sqrt{(bb - ff)(bb - nff)} - nffppqq$$

$$Af - pq = \frac{nfpq}{bb}$$

$$3Af - 3pq = \frac{3nfpq}{bb}.$$

n habebamus

$$fg - Af = \frac{nf^3}{bb}, \quad Ag - Af = 0,$$

$$Af + fg + gA - 3pq = \frac{3nfpq + nf}{bb}.$$

Quare ut arcus pq praecise sit triens totius peripheriae, necesse est

$$3pq = -ff \quad \text{seu} \quad pq = -\frac{1}{3}ff,$$

unde fit

$$pp + qq = ff - \frac{2ff}{3bb}\sqrt{(bb - ff)(bb - nff)} + \frac{nf^6}{9b^4}$$

hincque porro

$$qq \pm 2pq + pp = ff \pm \frac{2}{3}ff - \frac{2ff}{3bb}\sqrt{(bb - ff)(bb - nff)} + \frac{nf^6}{9b^4}$$

Fiet ergo

$$q - p = \frac{f}{3bb}\sqrt{(15b^4 + nf^4 - 6bb\sqrt{(bb - ff)(bb - nff)})},$$

$$q + p = \frac{f}{3bb}\sqrt{(3b^4 + nf^4 - 6bb\sqrt{(bb - ff)(bb - nff)})}.$$

Quia rectangulum $pq = -\frac{1}{3}ff$ est negativum, patet binarum absci
et q alteram esse positivam, alteram negativam. Cum autem sin
scissis bina curvae puncta respondeant, utrum conveniat, ex
$\sqrt{(bb - pp)}$ et $\sqrt{(bb - qq)}$, sive sint positivi sive negativi, dignoscitur
autem signa ita comparata esse oportet, ut satisfiat huic formulae

$$\sqrt{(bb - qq)} = \frac{b^3\sqrt{(bb - ff)(bb - pp)} - bfp\sqrt{(bb - nff)(bb - npp)}}{b^4 - nffpp}.$$

CASUS $n = \frac{1}{2}$

57. Prae ceteris hic casus $n = \frac{1}{2}$ seu $bb = 2aa$ est notatu dig
hoc solo radicale cubicum rationale evadit. Erit scilicet

$$\sqrt[3]{2nn(1-n)^2} = \frac{1}{2} \quad \text{et} \quad P = bb\sqrt{3};$$

unde

$$R + Q = 0 \quad \text{et} \quad R - Q = 2b^4\sqrt{3}$$

ideoque

$$Q = -b^4\sqrt{3} \quad \text{et} \quad R = +b^4\sqrt{3}.$$

$$ff = -P \pm \sqrt{(PP - 2Q)} \quad \text{et} \quad ff = +P \pm \sqrt{(PP - 2R)},$$

$$\frac{ff}{bb} = -\sqrt{3} \pm \sqrt{(3 + 2\sqrt{3})} \quad \text{et} \quad \frac{ff}{bb} = +\sqrt{3} \pm \sqrt{(3 - 2\sqrt{3})}.$$

rum quatuor valorum bini posteriores sunt imaginarii, priorum vero
itivus locum habet, ita ut sit

$$ff = bb\left(-\sqrt{3} + \sqrt{(3 + 2\sqrt{3})}\right),$$

a hinc $ff < bb$. Cum porro punctum f supra axem ellipsis CB existat,

$$\sqrt{(bb - ff)} = -b\sqrt{\left(1 + \sqrt{3} - \sqrt{(3 + 2\sqrt{3})}\right)}$$

$$\sqrt{(bb - nff)} = \frac{b}{\sqrt{2}}\sqrt{\left(2 + \sqrt{3} - \sqrt{(3 + 2\sqrt{3})}\right)},$$

o

$$\sqrt{(bb - ff)(bb - nff)} = \frac{-bb}{\sqrt{2}}\sqrt{\left(8 + 5\sqrt{3} - (3 + 2\sqrt{3})\sqrt{(3 + 2\sqrt{3})}\right)}$$

$$\sqrt{(bb - ff)(bb - nff)} = -\frac{1}{2}bb\left(\sqrt{(9 + 6\sqrt{3})} - 2 - \sqrt{3}\right).$$

a nunc sit

$$ff = bb\left(\sqrt{(3 + 2\sqrt{3})} - \sqrt{3}\right),$$

$$2pq = -\frac{2}{3}bb\left(\sqrt{(3 + 2\sqrt{3})} - \sqrt{3}\right)$$

$$pp + qq = +\frac{2}{3}bb\left(3 - \frac{1}{3}\sqrt{(9 + 6\sqrt{3})}\right),$$

quibus fit

$$(q + p)^2 = \frac{2}{3}bb\left(+3 + \sqrt{3} - \sqrt{(3 + 2\sqrt{3})} - \frac{1}{3}\sqrt{(9 + 6\sqrt{3})}\right),$$

$$(q - p)^2 = \frac{2}{3}bb\left(+3 - \sqrt{3} + \sqrt{(3 + 2\sqrt{3})} - \frac{1}{3}\sqrt{(9 + 6\sqrt{3})}\right)$$

radicibus extractis

$$q + p = \frac{1}{3}b\sqrt{(3 + \sqrt{3})(6 - 2\sqrt{(3 + 2\sqrt{3})})},$$

$$q - p = \frac{1}{3}b\sqrt{(3 - \sqrt{3})(6 + 2\sqrt{(3 + 2\sqrt{3})})}.$$

$$
\begin{array}{rlrl}
ff = & 0{,}8104090\,bb, & f = & 0{,}900227\ldots \\
\sqrt{(bb - ff)} = & 0{,}4354205\,b, & \sqrt{(bb - nff)} = & +\,0{,}771230\ldots \\
2pq = & 0{,}5102727\,bb, & (q + p)^2 = & 0{,}481134\ldots \\
pp + qq = & +\,1{,}0214069\,bb, & (q - p)^2 = & 1{,}561679\ldots \\
q + p = & 0{,}6936383\,b, & p = & 0{,}971654\ldots \\
p - q = & 1{,}2496712\,b, & q = & 0{,}278016\ldots
\end{array}
$$

quos valores pro p et q figura propemodum refert; atque ex for

$$\sqrt{(bb - pp)} \quad \text{et} \quad \sqrt{(bb - qq)}$$

involvente intelligitur punctum p infra axem AB, punctum q

cum capi debere.

CONSIDERATIO FORMULARUM QUARUM INTEGRATIO PER ARCUS SECTIONUM CONICARUM ABSOLVI POTEST

Commentatio 273 indicis Eneströmiani
Novi commentarii academiae scientiarum Petropolitanae 8 (1760/1), 1763, p. 129—14
Summarium ibidem p. 21—23

SUMMARIUM

Quando integrationes algebraice perficere non licet, valores integralium per qu
s linearum curvarum vulgo exhiberi solent, dum scilicet linea curva assignatur,
eundem valorem exprimat vel saltem eiusmodi quantitatem, ex qua is determinari p
r huiusmodi quantitates, quae, dum limites Algebrae communis quasi transcen
nscendentes appellantur, frequentissime occurrunt, quae a quadratura circuli et hyper
lent, quorsum omnes formulas integrales nullam irrationalitatem involventes reduci
stat, atque hae binae transcendentium species iam ita usu in Analysin sunt recept
emodum instar algebraicarum tractentur. Quae nimirum a quadratura circuli pen
nunc quidem per calculum angulorum felicissime expediuntur, quemadmodum eae,
uadratura hyperbolae pendent, logarithmis comprehendi solent, quorum calculus
inter elementa refertur. Quodsi vero quadraturis magis complicatis opus est, ev
lo maioribus difficultatibus est obnoxia. Etsi enim descriptio linearum curvarum
tur, tamen in praxi nimis est molestum areas iis inclusas satis exacte dimetiri.
causam iam pridem geometrae in hoc elaboraverunt, ut loco quadraturarum potius
tiones curvarum in hunc usum traducerent; quia, statim ac linea curva accurate
ripta, longitudinem cuiusque arcus sine ullo apparatu ope fili dimetiri licet, in
otio olim HERMANNUS[1]) immortalem gloriam est assecutus, dum problema ab alii

1) IAC. HERMANN, *Solutio propria duorum problematum geometricorum in Actis Erudit.*
s. Aug. a se propositorum, Acta erud. 1723, p. 171. A. K.

curvas adeo algebraicas invenire docuit, quarum rectificatione idem praesta
igitur nullum sit dubium, quin huiusmodi constructiones eo sint elegantiores
curvae, quarum rectificatio adhibetur, describi queant, in hoc negotio sect
Ellipsi scilicet et Hyperbolae, merito primae partes sunt tribuendae; et
difficillimum sit indolem earum formularum integralium perspicere, quarum v
sive ellipticos sive hyperbolicos exprimere liceat, Auctor hic singulari meth
formulas integrales investigat, quae hoc modo constructionem admittunt. Celeb
quidem hoc idem argumentum iam pridem in Actis Acad. Reg. Prussic
EULERI vero methodus plane nova, qua meras sectionum conicarum aliarumque
se comparare docuit, in hac investigatione eximiam praestitit utilitatem, u
multo uberius contulisse videatur. Plurimae autem transformationes, quibus
arcuum evolutione utitur, in Analysi haud spernendam utilitatem habere poterunt
ac dignitati huiusmodi investigationum nihil detrahetur, si observaverimus
calculi applicatione ad praxin neque curvarum quadraturam neque rectificatio
desiderari, cum omnia multo facilius et accuratius per methodos approxim
queant.

LEMMATA[2]

$$\text{I.}\quad \int dz\sqrt{\frac{f+gz}{h+kz}} = \frac{2}{k}\int dx\sqrt{\frac{fk-gh+gxx}{k}}$$

posito $x=\sqrt{(h+kz)}$.

$$\text{II.}\quad \int\frac{zz\,dz}{\sqrt{(f+gzz)(h+kzz)}} = \frac{1}{g}\int dx\sqrt{\frac{xx-f}{gh-fk+kxx}} = \frac{1}{k}\int dy\sqrt{\frac{yy-h}{fk-gh+gyy}}$$

posito $x=\sqrt{(f+gzz)}$ et $y=\sqrt{(h+kzz)}$.

1) J. d'ALEMBERT, *Recherches sur le calcul intégral. Seconde partie. De*
se rapportent à la rectification de l'ellipse ou de l'hyperbole, Mém. de l'acad. d
(1746), 1748, p. 200; *Suite des recherches sur le calcul intégral*, Mém. de l
Berlin, 4 (1748), 1750, p. 249. A. K.

2) Demonstrationes lemmatum et theorematum sequentium reperiuntur in
295 (indicis ENESTROEMIANI), quae sine dubio est huc ...; vide p. 256. A

$$\ldots\frac{\ldots gzz)}{\ldots zz)^{\frac{3}{2}}} = -\frac{1}{k}\int dx\sqrt{\frac{g+(fk-gh)xx}{1-hxx}} = \frac{1}{h}\int dy\sqrt{\frac{f+(gh-fk)yy}{1-kyy}}$$

posito $x = \frac{1}{\sqrt{(h+kzz)}}$ et $y = \frac{z}{\sqrt{(h+kzz)}}$.

$$\ldots\frac{\ldots kzz)}{\ldots zz)^{\frac{3}{2}}} = -\frac{1}{g}\int dx\sqrt{\frac{k+(gh-fk)xx}{1-fxx}} = \frac{1}{f}\int dy\sqrt{\frac{h+(fk-gh)yy}{1-gyy}}$$

posito $x = \frac{1}{\sqrt{(f+gzz)}}$ et $y = \frac{z}{\sqrt{(f+gzz)}}$.

$$\ldots\frac{dz}{\ldots^{\frac{3}{2}}\sqrt{(h+kzz)}} = \frac{1}{f}\int dx\sqrt{\frac{1-gxx}{h+(fk-gh)xx}} = \frac{1}{fk-gh}\int dy\sqrt{\frac{k-gyy}{fyy-h}}$$

posito $x = \frac{z}{\sqrt{(f+gzz)}}$ et $y = \sqrt{\frac{h+kzz}{f+gzz}}$.

$$\ldots\frac{dz}{\ldots^{\frac{3}{2}}\sqrt{(f+gzz)}} = \frac{1}{h}\int dx\sqrt{\frac{1-kxx}{f+(gh-fk)xx}} = \frac{1}{gh-fk}\int dy\sqrt{\frac{g-kyy}{hyy-f}}$$

posito $x = \frac{z}{\sqrt{(h+kzz)}}$ et $y = \sqrt{\frac{f+gzz}{h+kzz}}$.

$$\ldots\frac{zdz}{\ldots^{\frac{3}{2}}\sqrt{(h+kzz)}} = -\frac{1}{g}\int dx\sqrt{\frac{1-fxx}{k+(gh-fk)xx}} = \frac{1}{fk-gh}\int dy\sqrt{\frac{fyy-h}{k-gyy}}$$

posito $x = \frac{1}{\sqrt{(f+gzz)}}$ et $y = \sqrt{\frac{h+kzz}{f+gzz}}$.

$$\ldots\frac{zzdz}{\ldots)^{\frac{3}{2}}\sqrt{(f+gzz)}} = -\frac{1}{k}\int dx\sqrt{\frac{1-hxx}{g+(fk-gh)xx}} = -\frac{1}{gh-fk}\int dy\sqrt{\frac{hyy-f}{g-kyy}}$$

posito $x = \frac{1}{\sqrt{(h+kzz)}}$ et $y = \sqrt{\frac{f+gzz}{h+kzz}}$.

THEOREMATA

I. $\int dz\sqrt{\frac{f+gzz}{h+kzz}} = \frac{1}{k}\int dx\sqrt{\frac{fk-gh+gxx}{xx-h}}$

posito $x = \sqrt{(h+kzz)}$.

positio $x = \sqrt{\frac{f+gz}{h+kz}}$

III. $\int dz \sqrt{\frac{f+gzz}{h+kzz}} = z\sqrt{\frac{f+gzz}{h+kz}} + \frac{gh-fk}{k}\int dx \sqrt{\frac{1-hxx}{g+(fk-gh\ldots}}$

positio $x = \frac{1}{\sqrt{(h+kzz)}}$

IV. $\int dz \sqrt{\frac{f+gzz}{h+kzz}} = \frac{g}{k}z\sqrt{\frac{h+kzz}{f+gzz}} + \frac{fk-gh}{k}\int dx \sqrt{\frac{1-gxx}{h+(fk-gh\ldots}}$

positio $x = \frac{z}{\sqrt{(f+gzz)}}$

V. $\int dz \sqrt{\frac{f+gzz}{h+kzz}} = \frac{g}{k}z\sqrt{\frac{h+kz}{f+gzz}} + \frac{f}{k}\int dx \sqrt{\frac{k-gxx}{fxx-h}}$

positio $x = \sqrt{\frac{h+kzz}{f+gzz}}$

VI. $\int dz \sqrt{\frac{f+gzz}{h+kzz}} = \frac{f}{h}\int dz \sqrt{\frac{h+kzz}{f+gzz}} + \frac{gh-fk}{gh}\int dx \sqrt{\frac{xx}{gh-fk\ldots}}$

positio $x = \sqrt{(f+gzz)}$.

VII. $\int dz \sqrt{\frac{f+gzz}{h+kzz}} = \frac{f}{h}\int dz \sqrt{\frac{h+kzz}{f+gzz}} + \frac{gh-fk}{hk}\int dx \sqrt{\frac{xx}{fk-gh\ldots}}$

positio $x = \sqrt{(h+kzz)}$.

VIII. $\int dz \sqrt{\frac{f+gzz}{h+kzz}} = z\sqrt{\frac{f+gzz}{h+kzz}} + P + Q$,

ubi

$$P = \frac{gh-fk}{gk}\int dx \sqrt{\frac{g+(fk-gh)xx}{1-hxx}} = \frac{fk-gh}{gh}\int dy \sqrt{\frac{f+(gh-f\ldots}{1-kyy\ldots}}$$

positio $x = \frac{1}{\sqrt{(h+kzz)}}$ et $y = \frac{z}{\sqrt{(h+kzz)}}$

et

$$Q = \frac{f(fk-gh)}{gh}\int dx \sqrt{\frac{1-kxx}{f+(gh-fk)xx}} = \frac{f}{g}\int dy \sqrt{\frac{g-kyy}{kyy-f\ldots}}$$

positio $x = \frac{z}{\sqrt{(h+kzz)}}$ et $y = \sqrt{\frac{f+gzz}{h+kzz}}$.

bi

$$P = \frac{gh-fk}{gh}\int dx\sqrt{\frac{xx-f}{gh-fk+kxx}} = \frac{gh-fk}{hk}\int dy\sqrt{\frac{yy-h}{fk-gh+gyy}}$$

posito $x = \sqrt{(f+gzz)}$ et $y = \sqrt{(h+kzz)}$

atque

$$Q = \frac{f(gh-fk)}{gh}\int dx\sqrt{\frac{1-kxx}{f+(gh-fk)xx}} = \frac{f}{g}\int dy\sqrt{\frac{g-kyy}{kyy-f}}$$

posito $x = \frac{z}{\sqrt{(h+kzz)}}$ et $y = \sqrt{\frac{f+gzz}{h+kzz}}$.

$$\text{X.}\quad \int dz\sqrt{\frac{f+gzz}{h+kzz}} = \frac{gh-fk}{gh}z\sqrt{\frac{f+gzz}{h+kzz}} + \frac{f}{h}\int dz\sqrt{\frac{h+kzz}{f+gzz}} + P,$$

ubi

$$P = \frac{gh-fk}{gk}\int dx\sqrt{\frac{g+(fk-gh)xx}{1-hxx}} = \frac{fk-gh}{gh}\int dy\sqrt{\frac{f+(gh-fk)yy}{1-kyy}}$$

posito $x = \frac{1}{\sqrt{(h+kzz)}}$ et $y = \frac{z}{\sqrt{(h+kzz)}}$.

$$\text{XI.}\quad \int dz\sqrt{\frac{f+gzz}{h+kzz}} = \frac{f}{h}z\sqrt{\frac{h+kzz}{f+gzz}} + P + Q,$$

ubi

$$P = \frac{gh-fk}{gh}\int dx\sqrt{\frac{xx-f}{gh-fk+kxx}} = \frac{gh-fk}{hk}\int dy\sqrt{\frac{yy-h}{fk-gh+gyy}}$$

posito $x = \sqrt{(f+gzz)}$ et $y = \sqrt{(h+kzz)}$

atque

$$Q = \frac{f(fk-gh)}{gh}\int dx\sqrt{\frac{1-fxx}{k+(gh-fk)xx}} = \frac{-f}{h}\int dy\sqrt{\frac{fyy-h}{k-gyy}}$$

posito $x = \frac{1}{\sqrt{(f+gzz)}}$ et $y = \sqrt{\frac{h+kzz}{f+gzz}}$.

$$\text{XII.}\quad \int dz\sqrt{\frac{f+gzz}{h+kzz}} = \frac{g}{k}z\sqrt{\frac{h+kzz}{f+gzz}} + P + Q,$$

ubi

$$P = \frac{f(gh-fk)}{ghk}\int dx\sqrt{\frac{k+(gh-fk)xx}{1-fxx}} = \frac{fk-gh}{hk}\int dy\sqrt{\frac{h+(fk-gh)yy}{1-gyy}}$$

posito $x = \frac{1}{\sqrt{(f+gzz)}}$ et $y = \sqrt{\frac{z}{f+gzz}}$

$$Q = \frac{f(fk-gh)}{gh}\int dx \sqrt{\frac{1 \ldots fxx}{k+(gh-fk)xx}} \ldots \frac{f}{h}\int dy \sqrt{\frac{fyy \ldots h}{k \ldots}} \ldots$$

$$\text{posito } x = \frac{1}{\sqrt{(f+gzz)}} \quad \text{et} \quad y = \sqrt{\frac{h+kzz}{f+gzz}}.$$

$$\text{XIII. } \int dz \sqrt{\frac{f+gzz}{h+kzz}} = \frac{gh-fk}{hk}\ldots\sqrt{\frac{h+kzz}{f+gzz}} + \frac{f}{h}\int dz \sqrt{\frac{h+kzz}{f+gzz}} \ldots$$

ubi

$$p = \frac{f(gh-fk)}{ghk}\int dx \sqrt{\frac{k+(gh-fk)xx}{1-fxx}} \ldots \frac{fk-gh}{hk}\int dy \sqrt{\frac{h+(fk \ldots}{1- \ldots}} \ldots$$

$$\text{posito } x = \frac{1}{\sqrt{(f+gzz)}} \quad \text{et} \quad y = \ldots\frac{\ldots}{\sqrt{(f+gzz)}}.$$

THEOREMA SINGULARE[1]

$$\int dz \sqrt{\frac{f+gzz}{h+kzz}} - \frac{gxz}{\sqrt{p}} = \int dx \sqrt{\frac{f+gxx}{h+kxx}},$$

ubi p denotat constantem arbitrariam, posita inter x et z hac relatione

$$gkxxzz - pxx - pzz - 2xz\sqrt{p(f \ldots p+fk)(p+gh) \ldots fh} = 0$$

sive

$$x = \frac{z\sqrt{(p+fk)(p+gh)} \ldots \sqrt{p(f+gzz)(h+kzz)}}{p-gkzz}.$$

HYPOTHESIS

Haec scribendi formula $\Pi x[a]$ denotet sectionis conicae, cuius semiaxis … $=1$ et semiaxis transversus $=a$, arcum a vertice sumtum, cui in a… conveniat abscissa $= x$.

COROLLARIUM

… sit quantitas positiva, hoc modo designatur arcus … arcus hyperbolae, si modo x fuerit quantitas positi…

[1] … ommentationes 261 et 264 (indicis Enestroemiani); vide p. 153 …

Casus I $\int dz\sqrt{\frac{f+gzz}{h-kzz}}$

Integrale est immediate

$$C-\frac{fk+gh}{k\sqrt{fk}}\Pi\frac{-fk}{fk+gh}\left(1-z\sqrt{\frac{k}{h}}\right)\left[\frac{-fk}{fk+gh}\right]$$

vel etiam per theor. I

$$C+\frac{f}{\sqrt{(fk+gh)}}\Pi\frac{fk+gh}{fk}\left(1-\frac{\sqrt{(h-kzz)}}{\sqrt{h}}\right)\left[\frac{fk+gh}{fk}\right].$$

Casus II $\int dz\sqrt{\frac{f-gzz}{h-kzz}}$ existente $fk>gh$

Integrale est immediate

$$C-\frac{fk-gh}{k\sqrt{fk}}\Pi\frac{fk}{fk-gh}\left(1-z\sqrt{\frac{k}{h}}\right)\left[\frac{fk}{fk-gh}\right]$$

vel etiam per theor. I

$$C+\frac{f}{\sqrt{(fk-gh)}}\Pi\frac{fk-gh}{fk}\left(1-\frac{\sqrt{(h-kzz)}}{\sqrt{h}}\right)\left[\frac{fk-gh}{fk}\right].$$

Casus III $\int dz\sqrt{\frac{-f+gzz}{-h+kzz}}$ existente $fk<gh$

Integrale est immediate

$$C+\frac{gh-fk}{k\sqrt{fk}}\Pi\frac{fk}{gh-fk}\left(z\sqrt{\frac{k}{h}}-1\right)\left[\frac{-fk}{gh-fk}\right].$$

Casus IV $\int dz\sqrt{\frac{f+gzz}{h+kzz}}$ existente $fk<gh$

Integrale est per theor. I

$$C+\frac{f}{\sqrt{(gh-fk)}}\Pi\frac{gh-fk}{fk}\left(\frac{\sqrt{(h+kzz)}}{\sqrt{h}}-1\right)\left[\frac{-gh+fk}{fk}\right].$$

Casus V $\int dz\sqrt{\frac{-f+gzz}{h+kzz}}$

Integrale est per theor. III

$$C+z\sqrt{\frac{-f+gzz}{h+kzz}}-\frac{f}{\sqrt{(fk+gh)}}\Pi\frac{fk+gh}{fk}\left(1-\frac{\sqrt{(fk+gh)}}{\sqrt{g(h+kzz)}}\right)\left[\frac{fk+}{f}\right.$$

vel etiam per theor. II

$$C + z\sqrt{\frac{f+gzz}{h+kzz}} + \frac{fk+gh}{k\sqrt{fk}}\, Il\, \frac{fk}{fk+gh}\left(1 \ldots \frac{+k(f+gzz)}{+g(h+kzz)}\right.$$ [illegible]

Casus VI $\int dz \sqrt{\frac{f+gzz}{h+kzz}}$ existente $fk \ldots gh$

Integrale est per theor. III

$$C + z\sqrt{\frac{f+gzz}{h+kzz}} \ldots \frac{f}{\sqrt{(fk-gh)}}\, Il\, \frac{fk-gh}{fk}\left(1 \ldots \frac{+(fk-gh)}{+g(h+kzz)}\right.$$ [illegible]

vel etiam per theor. II

$$C + z\sqrt{\frac{f+gzz}{h+kzz}} + \frac{fk-gh}{k\sqrt{fk}}\, Il\, \frac{fk}{fk-gh}\left(1 \ldots \frac{+k(f+gzz)}{+g(h+kzz)}\right.$$ [illegible]

Casus VII $\int dz \sqrt{\frac{f-gzz}{h-kzz}}$ existente $fk \ldots gh$

Integrale est per theor. III

$$C + \frac{gz}{k}\sqrt{\frac{h-kzz}{f-gzz}} \ldots \frac{gh-fk}{k\sqrt{fk}}\, Il\, \frac{fk}{gh-fk}\left(\frac{+f(h-kzz)}{+h(f-gzz)} \ldots 1\right)$$

Casus VIII $\int dz \sqrt{\frac{f+gzz}{h-kzz}}$ existente $fk \ldots gh$

Integrale est per theor. II

$$C + z\sqrt{\frac{f+gzz}{h-kzz}} \ldots \frac{f}{\sqrt{(gh-fk)}}\, Il\, \frac{gh-fk}{fk}\left(\frac{\sqrt{(gh-fk)}}{+g(h-kzz)} \ldots 1\right.$$ [illegible]

vel etiam per theor. V

$$C \ldots \frac{gz}{k}\sqrt{\frac{h-kzz}{f+gzz}} + \frac{f}{\sqrt{(gh-fk)}}\, Il\, \frac{gh-fk}{fk}\left(\frac{z+(gh-fk)}{+h(f+gzz)} \ldots 1\right.$$ [illegible]

Casus IX $\int dz \sqrt{\frac{f+gzz}{h+kzz}}$ existente $fk \ldots gh$

Integrale est per theor. X

$$C \ldots \frac{(fk-gh)z}{gh}\sqrt{\frac{f+gzz}{h+kzz}} \ldots \frac{fk-gh}{k\sqrt{fk}}\, Il\, \frac{fk}{gh}\left(1 \ldots \frac{+k}{+(h+k}\right.$$ [illegible]

$$+ \frac{f}{\sqrt{(fk-gh)}}\, Il\, \frac{fk-gh}{gh}\left(\frac{\sqrt{(f+gzz)}}{\sqrt{f}} \ldots 1\right)\left[\frac{fk+g}{gh}\right.$$ [illegible]

el etiam per theor. XIII

$$C-\frac{(fk-gh)z}{hk}\sqrt{\frac{h+kzz}{f+gzz}}+\frac{fk-gh}{k\sqrt{fk}}\,\Pi^{\frac{fk}{gh}}\Big(1-\frac{\sqrt{f}}{\sqrt{(f+gzz)}}\Big)\Big[\frac{fk}{gh}\Big]$$

$$+\frac{f}{\sqrt{(fk-gh)}}\,\Pi^{\frac{fk-gh}{gh}}\Big(\frac{\sqrt{(f+gzz)}}{\sqrt{f}}-1\Big)\Big[\frac{-fk+gh}{gh}\Big]^{1)}.$$

Casus X $\int dz\sqrt{\frac{f-gzz}{-h+kzz}}$ existente $fk>gh$

Integrale est per theor. IX

$$C+\frac{fkz}{gh}\sqrt{\frac{f-gzz}{-h+kzz}}+\frac{fk-gh}{k\sqrt{fk}}\,\Pi^{\frac{fk}{gh}}\Big(1-\frac{\sqrt{k(f-gzz)}}{\sqrt{(fk-gh)}}\Big)\Big[\frac{fk}{gh}\Big]$$

$$-\frac{f}{\sqrt{(fk-gh)}}\,\Pi^{\frac{fk-gh}{gh}}\Big(\frac{z\sqrt{(fk-gh)}}{\sqrt{f(-h+kzz)}}-1\Big)\Big[\frac{-fk+gh}{gh}\Big]$$

vel etiam per theor. XI

$$C-\frac{fz}{h}\sqrt{\frac{-h+kzz}{f-gzz}}+\frac{fk-gh}{k\sqrt{fk}}\,\Pi^{\frac{fk}{gh}}\Big(1-\frac{\sqrt{k(f-gzz)}}{\sqrt{(fk-gh)}}\Big)\Big[\frac{fk}{gh}\Big]$$

$$+\frac{f}{\sqrt{(fk-gh)}}\,\Pi^{\frac{fk-gh}{gh}}\Big(\frac{\sqrt{(fk-gh)}}{\sqrt{k(f-gzz)}}-1\Big)\Big[\frac{-fk+gh}{gh}\Big].$$

Casus XI $\int dz\sqrt{\frac{f+gzz}{-h+kzz}}$

Integrale est per theor. XI

$$C-\frac{fz}{h}\sqrt{\frac{-h+kzz}{f+gzz}}+\frac{f}{\sqrt{(fk+gh)}}\,\Pi^{\frac{fk+gh}{gh}}\Big(1-\frac{\sqrt{(fk+gh)}}{\sqrt{k(f+gzz)}}\Big)\Big[\frac{fk+gh}{gh}\Big]$$

$$+\frac{fk+gh}{k\sqrt{fk}}\,\Pi^{\frac{fk}{gh}}\Big(\frac{\sqrt{k(f+gzz)}}{\sqrt{(fk+gh)}}-1\Big)\Big[\frac{-fk}{gh}\Big]$$

vel etiam per theor. XII

$$C+\frac{gz}{k}\sqrt{\frac{-h+kzz}{f+gzz}}+\frac{f}{\sqrt{(fk+gh)}}\,\Pi^{\frac{fk+gh}{gh}}\Big(1-\frac{\sqrt{(fk+gh)}}{\sqrt{k(f+gzz)}}\Big)\Big[\frac{fk+gh}{gh}\Big]$$

$$+\frac{fk+gh}{k\sqrt{fk}}\,\Pi^{\frac{fk}{gh}}\Big(\frac{\sqrt{f}}{\sqrt{(f+gzz)}}-1\Big)\Big[\frac{-fk}{gh}\Big].$$

1) Editio princeps: $+\frac{f}{\sqrt{(fh-gh)}}\,\Pi^{\frac{fk-gh}{gh}}\Big(\frac{\sqrt{(h+kzz)}}{\sqrt{h}}-1\Big)\Big[\frac{-fk+gh}{gh}\Big]$.

Correxit A. K.

Integrale est per theor. XIII

$$C - \frac{(fk+gh)}{hk} z \sqrt{\frac{h+kzz}{f+gzz}} + \frac{f}{\sqrt{(fk+gh)}} \Pi^{\frac{fk+gh}{gh}} \left(1 - \frac{\sqrt{(f+gzz)}}{\sqrt{f}} \right) \Big| \frac{f}{\cdots}$$

$$+ \frac{fk+gh}{k\sqrt{fk}} \Pi^{\frac{fk}{gh}} \left(\frac{\sqrt{f}}{\sqrt{(f+gzz)}} - 1 \right) \Big| \frac{fk}{gh} \Big| .$$

Omnes ergo casus formulae

$$\int dx \sqrt{\frac{\alpha+\beta xx}{\gamma+\delta xx}},$$

quomodocumque litterae α, β, γ, δ fuerint comparatae, per arcus conicarum integrari possunt.

Non solum igitur formulae initio commemoratae integrationem sectionum conicarum admittunt, sed etiam innumerabiles aliae, quae substitutionem ad formam

$$\int dz \sqrt{\frac{\alpha+\beta zz}{\gamma+\delta zz}}$$

se reduci patiuntur, cuiusmodi sunt

$$1.\ \int \frac{dz}{zz} \sqrt{\frac{f+gzz}{h+kzz}} = -\int dx \sqrt{\frac{fxx+g}{hxx+k}} = \frac{1}{h} \int dy \sqrt{\frac{fyy - fk + \cdots}{yy - k}}$$

$$\text{posito } x = \frac{1}{z} \text{ et } y = \frac{\sqrt{(h+kzz)}}{z},$$

$$2.\ \int \frac{dz}{zz\sqrt{(f+gzz)(h+kzz)}} = -\frac{1}{f} \int dx \sqrt{\frac{xx - g}{hxx + fk - gh}} = -\frac{1}{h} \int dy \sqrt{\cdots}$$

$$\text{posito } x = \frac{\sqrt{(f+gzz)}}{z} \text{ et } y = \frac{\sqrt{(h+kzz)}}{z},$$

$$3.\ \int \frac{dz}{\sqrt{(f+gzz)(h+kzz)}} = \frac{k}{fk-gh} \int dz \sqrt{\frac{f+gzz}{h+kzz}} - \frac{g}{fk-gh} \int dz \sqrt{\frac{h+\cdots}{f+\cdots}}$$

cuius formulae reductio etiam ita instituitur

$$\int \frac{dz}{\sqrt{(f+gzz)(h+kzz)}} = \frac{f}{fk-gh} \int dx \sqrt{\frac{k-gxx}{fxx-h}} + \frac{g}{fk-gh} \int dx \sqrt{\frac{f\cdots}{k\cdots}}$$

$$\text{posito } x = \sqrt{\frac{h+kzz}{f+gzz}}$$

m sic

$$\int\frac{dz}{\sqrt{(f+gzz)(h+kzz)}}=\int dx\sqrt{\frac{1-gxx}{h+(fk-gh)xx}}=\int dy\sqrt{\frac{1-fyy}{k+(gh-fk)yy}}$$

$$\text{posito } x=\frac{z}{\sqrt{(f+gzz)}} \quad\text{et}\quad y=\frac{1}{\sqrt{(f+gzz)}}.$$

namus $zz=v$ atque obtinebimus sequentes formulas, quae pariter per ctionum conicarum construi poterunt

$$1.\ \int\frac{dv\sqrt{(f+gv)}}{\sqrt{v(h+kv)}} \qquad 2.\ \int\frac{dv\sqrt{(f+gv)}}{v\sqrt{v(h+kv)}}$$

$$3.\ \int\frac{dv\sqrt{v}}{\sqrt{(f+gv)(h+kv)}} \qquad 4.\ \int\frac{dv}{\sqrt{v(f+gv)(h+kv)}}$$

$$5.\ \int\frac{dv\sqrt{(f+gv)}}{(h+kv)^{\frac{3}{2}}\sqrt{v}} \qquad 6.\ \int\frac{dv}{v\sqrt{v(f+gv)(h+kv)}}$$

$$7.\ \int\frac{dv}{(f+gv)^{\frac{3}{2}}\sqrt{v(h+kv)}} \qquad 8.\ \int\frac{dv\sqrt{v}}{(f+gv)^{\frac{3}{2}}\sqrt{(h+kv)}};$$

n vicissim posito $v=zz$ ad formas praecedentes reducuntur.

nc patet istam formulam satis late patentem ad arcus sectionum coni-oduci posse

$$\int\frac{(A+Bu)du}{\sqrt{(\alpha+\beta u)(\gamma+\delta u)(\varepsilon+\zeta u)}},$$

nprimis notari meretur. Ponatur enim $\alpha+\beta u=v$, ut sit $u=\frac{v-\alpha}{\beta}$, formula transmutabitur in hanc

$$\int\frac{dv(A\beta-B\alpha+Bv)}{\beta\sqrt{v(\beta\gamma-\alpha\delta+\delta v)(\beta\varepsilon-\alpha\zeta+\zeta v)}},$$

binas formulas sub no. 3 et 4 allatas revocatur. Quare si

$$\alpha+\beta x+\gamma xx+\delta x^3$$

tres factores reales, haec formula

$$\int\frac{dx(A+Bx)}{\sqrt{(\alpha+\beta x+\gamma xx+\delta x^3)}}$$

xposito integrari poterit; semper autem unum factorem certe habet

referri potest $y(pp+2npqy+qqyy)$ existente $nn<1$. Hinc et integrale harum formularum

$$\int\frac{C\,dy}{\sqrt{y(pp+2npqy+qqyy)}}+\int\frac{D\,dy\sqrt{y}}{\sqrt{(pp+2npqy+qqyy)}}.$$

Ponatur

$$\sqrt{(pp+2npqy+qqyy)}=p+qyz$$

fietque $y=\frac{2p(z-n)}{q(1-zz)}$, qua substitutione prior formula abit in

$$\frac{C\sqrt{2}}{\sqrt{pq}}\int\frac{dz}{\sqrt{(z-n)(1-zz)}}$$

constructibilem, posterior vero in hanc

$$\frac{2D\sqrt{2p}}{q\sqrt{q}}\int\frac{dz\sqrt{(z-n)}}{(1-zz)^{\frac{3}{2}}};$$

cum vero sit

$$\int\frac{dz\sqrt{(z-n)}}{(1-zz)^{\frac{3}{2}}}=\frac{\sqrt{(1-nz)}}{\sqrt{(1-zz)}}\dots-\frac{1}{2}\int\frac{dz}{\sqrt{(z-n)(1-zz)}}\dots,$$

etiam haec per superiora construi potest. Atque in genere habetur huius formulae

$$\int\frac{dx(A+Bx)}{\sqrt{(\alpha+\beta x+\gamma xx+\delta x^3)}}$$

PROBLEMA 1

Integrationem huius formulae

$$\int\frac{dx}{\sqrt{(a+bx+cxx+dx^3+ex^4)}}$$

per arcus sectionum conicarum perficere.

SOLUTIO

Quantitatem $a+bx+cxx+dx^3+ex^4$ semper in duos trinomiales reales resolvere licet, qui sint $\alpha+2\beta x+\gamma xx$ et δ

$$\int \frac{dx}{\sqrt{(\alpha+2\beta x+\gamma xx)(\delta+2\varepsilon x+\zeta xx)}}.$$

$$\delta+2\varepsilon x+\zeta xx=(\alpha+2\beta x+\gamma xx)y,$$

la proposita fiat

$$\int \frac{dx}{(\alpha+2\beta x+\gamma xx)\sqrt{y}}.$$

tio assumta per radicis extractionem praebet

$$\varepsilon+\zeta x-\beta y-\gamma xy=\sqrt{(pyy+qy+r)}$$

$$p=\beta\beta-\alpha\gamma,\quad q=\alpha\zeta-2\beta\varepsilon+\gamma\delta\quad \text{et}\quad r=\varepsilon\varepsilon-\delta\zeta.$$

o eadem differentiata dat

$$dx(\varepsilon+\zeta x-\beta y-\gamma xy)=\frac{1}{2}dy(\alpha+2\beta x+\gamma xx)$$

$$\frac{dx}{\alpha+2\beta x+\gamma xx}=\frac{\frac{1}{2}dy}{\varepsilon+\zeta x-\beta y-\gamma xy}.$$

pro hoc postremo denominatore valorem irrationalem modo inventu
mus, formula proposita abit in hanc

$$\int \frac{\frac{1}{2}dy}{\sqrt{y(pyy+qy+r)}},$$

egratio per arcus sectionum conicarum supra est ostensa.
igitur nascitur quaestio, quid tenendum sit de hac formula

$$\int \frac{dx(A+Bx+Cxx)}{\sqrt{(a+bx+cxx+dx^3+ex^4)}},$$

enim est non necesse esse, ut numeratori altiores potestates ipsi
tur; quam etiam Cel. D'ALEMBERT[1]) fatetur se in genere ad rect
sectionum conicarum perducere non posse. Considerat quidem i

---

de notam 1 p. 236. A. K.

ita ut formula sit

$$\int \frac{dx\sqrt{z}}{\sqrt{(b+cx+dx^2+ex^3)}}$$

conaturque ostendere (p. 257) eius integrationem casu ad ... sectionum conicarum absolvi posse; verum methodus, qua ... minime conficere videtur, uti rem accuratius perpendenti mox ... formationes autem, quas deinceps tradit, casus nonnunquam ... biles suppeditant. Quocirca haec investigatio, uti est difficilli... attentione digna est censenda, unde etiam mea tentamina ... stioni proposuisse iuvabit.

PROBLEMA 2

Investigare conditiones, sub quibus integrationem huius form...

$$\int \frac{dy(\mathfrak{P}+\mathfrak{Q}y+\mathfrak{R}yy)}{\sqrt{(\mathfrak{A}y^4+2\mathfrak{B}y^3+\mathfrak{C}yy+2\mathfrak{D}y+\mathfrak{E})}}$$

ad hanc simpliciorem

$$\int \frac{dx(P+Qx+Rxx)}{\sqrt{(Ax^4+Cxx+E)}}$$

reducere liceat.

SOLUTIO

Statuatur inter variabiles x et y talis relatio

$$\alpha xxyy + 2xy(\beta x + \gamma y) + \delta xx + \varepsilon yy + 2\zeta xy + 2\eta x + 2\theta y + \varkappa = 0,$$

cuius coefficientes ita determinentur, ut sit

$$\beta\zeta - \alpha\eta - \gamma\delta = 0, \qquad \zeta\theta - \gamma\varkappa - \varepsilon\eta = 0,$$
$$\gamma\gamma - \alpha\varepsilon = \mathfrak{A}, \qquad \gamma\zeta - \alpha\theta - \beta\varepsilon = \mathfrak{B},$$
$$\eta\eta - \delta\varkappa = \mathfrak{E}, \qquad \zeta\eta - \beta\varkappa - \delta\theta = \mathfrak{D}$$

et

$$\zeta\zeta + 2\gamma\eta - \alpha\varkappa - \delta\varepsilon - 4\beta\theta = \mathfrak{C},$$

hincque erit pro denominatore transformatae

$$A = \beta\beta - \alpha\delta, \quad E = \theta\theta - \varepsilon\varkappa$$

et

$$C = \zeta\zeta + 2\beta\theta - \alpha\varkappa - \delta\varepsilon - 4\gamma\eta.$$

... novem indefinitae litterae $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta, \varkappa$, his septem
...ons praescriptis utique satisfieri poterit relinqueturque adhuc una
...ostro determinanda. Si iam brevitatis gratia ponamus

$$\ldots y^4 + 2\mathfrak{B}y^3 + \mathfrak{C}y^2 + 2\mathfrak{D}y + \mathfrak{E} = Y \quad \text{et} \quad Ax^4 + Cxx + E = X,$$

...aequationis assumtae praebet

$$\alpha xyy + 2\beta xy + \delta x + \gamma yy + \zeta y + \eta = \sqrt{Y},$$
$$\alpha xxy + 2\gamma xy + \varepsilon y + \beta xx + \zeta x + \theta = \sqrt{X}$$

...fferentiatio ducit ad hanc aequationem

$$\frac{dy}{\sqrt{Y}} + \frac{dx}{\sqrt{X}} = 0.$$

...ergo

$$\int \frac{dy(\mathfrak{P} + \mathfrak{Q}y + \mathfrak{R}yy)}{\sqrt{(\mathfrak{A}y^4 + 2\mathfrak{B}y^3 + \mathfrak{C}y^2 + 2\mathfrak{D}y + \mathfrak{E})}} = V - \int \frac{dx(P + Qx + Rxx)}{\sqrt{(Ax^4 + Cx^2 + E)}}$$

...talis functio algebraica

$$V = mx + ny + pxy + \frac{1}{2}qxx + \frac{1}{2}ryy + txyy.$$

...tis differentialibus terminisque homogeneis seorsim aequatis reperien-
...ntes determinationes

$$m = \frac{\beta\mathfrak{R}}{\mathfrak{A}}, \quad n = \frac{\gamma\mathfrak{R}}{\mathfrak{A}}, \quad p = \frac{\alpha\mathfrak{R}}{\mathfrak{A}}, \quad q = 0, \quad r = 0 \quad \text{et} \quad t = 0,$$

...vero haec determinatio accedit, ut sit $\mathfrak{A}\mathfrak{Q} = \mathfrak{B}\mathfrak{R}$. Deinde vero fit

$$P = \mathfrak{P} + \frac{(\beta\theta - \gamma\eta)\mathfrak{R}}{\mathfrak{A}}, \quad Q = 0 \quad \text{et} \quad R = \frac{A\mathfrak{R}}{\mathfrak{A}}.$$

...ergo coefficientibus $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta, \varkappa$, quibus constat relatio
... y, ex iis innotescunt quantitates A, C, E, quibus inventis, si fuerit
...$\mathfrak{R}$, erit

$$\int \frac{dy(\mathfrak{P} + \mathfrak{Q}y + \mathfrak{R}yy)}{\sqrt{(\mathfrak{A}y^4 + 2\mathfrak{B}y^3 + \mathfrak{C}y^2 + 2\mathfrak{D}y + \mathfrak{E})}} = \text{Const.} + \frac{\mathfrak{R}}{\mathfrak{A}}(\beta x + \gamma y + \alpha xy)$$
$$- \int \frac{dx\left(\mathfrak{P} + \frac{(\beta\theta - \gamma\eta)\mathfrak{R}}{\mathfrak{A}} + \frac{A\mathfrak{R}}{\mathfrak{A}}xx\right)}{\sqrt{(Ax^4 + Cx^2 + E)}}$$

ad hanc simpliciorem

$$\int \frac{dx(P+Rxx)}{\sqrt{(Ax^4+Cx^2+E)}}.$$

COROLLARIUM 1

Determinatio coefficientium α, β, γ etc. commode ... sequenti modo instituetur. Primo quaeratur valor ipsius s ex hac aeq[uatione]

$$\mathfrak{C} = \frac{\mathfrak{A}\mathfrak{B}}{\mathfrak{A}} - \frac{\mathfrak{F}\mathfrak{F}ss}{\mathfrak{G}ss} + \frac{2\mathfrak{A}s}{\mathfrak{A}} - \frac{2\mathfrak{A}\mathfrak{G}}{\mathfrak{F}} \ldots,$$

quae cum sit cubica, certe valorem realem pro s ... que ad arbitrium quantitate t sit brevitatis gratia ... valores omnium 9 coefficientium ita se habebunt

$$\zeta = a\sqrt{\frac{\mathfrak{A}\mathfrak{A} - 3\mathfrak{A}\mathfrak{F}s + 3\mathfrak{A}\mathfrak{G}ss - \mathfrak{F}\mathfrak{G}\ldots}{ss \quad aa(\mathfrak{A} - \mathfrak{F}s)}}$$

$$\gamma = \frac{\zeta\eta}{2a^3}, \quad \alpha = \frac{a}{2t(s-aa)^2},$$

$$\eta = \frac{\zeta}{2a^3}, \quad \delta = \frac{a}{2st(s-aa)^2},$$

$$\beta = \frac{1}{2t(s-aa)^2}, \quad \theta = \frac{1}{2}t(2\mathfrak{A} + \mathfrak{G}s \ldots), \quad \ldots \frac{3}{a}$$

$$\iota = \frac{1}{2}t(4\mathfrak{A}a - 3\mathfrak{B}s + \mathfrak{D}aa), \quad \lambda = \frac{1}{2}t\,\mathfrak{G}sa \ldots$$

COROLLARIUM 2

Alio adhuc modo idem praestari potest. Extrac[ta] s ex hac aequatione

$$\mathfrak{C} = \frac{\mathfrak{A}\mathfrak{B}}{\mathfrak{A}} - \frac{\mathfrak{D}\mathfrak{D}as}{\mathfrak{G}ss} + \frac{2\mathfrak{A}\mathfrak{F}}{\mathfrak{A}} - \frac{2\mathfrak{A}\mathfrak{G}}{\mathfrak{F}s} \ldots$$

positoque brevitatis gratia $\frac{\mathfrak{A}}{\mathfrak{A}} - \frac{\mathfrak{G}aa}{\mathfrak{F}a} = n$ et sumto t pr[o] ...

1) Editio princeps: $\zeta = a\sqrt{\frac{\mathfrak{A}\mathfrak{A} - 3\mathfrak{A}\mathfrak{F}s + 3\mathfrak{A}\mathfrak{G}ss - \mathfrak{F}\mathfrak{G}s^3}{s \quad aa}}$.

$$\alpha = -\frac{1}{4tu}, \quad \beta = 0, \quad \gamma = \frac{1}{2}\sqrt{\frac{s(\mathfrak{B}+\mathfrak{D}s)}{u}}, \quad \delta = \frac{1}{4tsu},$$

$$= t(4\mathfrak{A}u - \mathfrak{B}s - \mathfrak{D}ss), \quad \zeta = \sqrt{\frac{u(\mathfrak{B}+\mathfrak{D}s)}{s}}, \quad \eta = \frac{1}{2}\sqrt{\frac{\mathfrak{B}+\mathfrak{D}s}{us}},$$

$$\theta = 2tu(\mathfrak{B}-\mathfrak{D}s)^{1)}, \quad \varkappa = t(\mathfrak{B}+\mathfrak{D}s - 4\mathfrak{C}su).$$

COROLLARIUM 3

erit $\mathfrak{A}:\mathfrak{E} = \mathfrak{B}\mathfrak{B}:\mathfrak{D}\mathfrak{D}$, aequatio cubica valori s definiendo fit inepta
incommodum facile tollitur transformanda formula differentiali per
$y = y \pm a$; qua etiam forma numeratoris non turbatur.

SCHOLION

$\mathfrak{N} = n\mathfrak{A}$ et $\mathfrak{Q} = n\mathfrak{B}$ integratio huius formulae

$$\int \frac{dy(\mathfrak{P} + n\mathfrak{B}y + n\mathfrak{A}yy)}{\sqrt{(\mathfrak{A}y^4 + 2\mathfrak{B}y^3 + \mathfrak{C}y^2 + 2\mathfrak{D}y + \mathfrak{E})}}$$

duci potest ad integrationem talis

$$\int \frac{dx(P + Rxx)}{\sqrt{(Ax^4 + Cxx + E)}},$$

denominator $Ax^4 + Cxx + E$ in huiusmodi duos factores reale
$h + kxx)$ se resolvi patitur, per rectificationem sectionum conicarum
at si talis resolutio non succedit, sequenti artificio negotium ab
rit.

PROBLEMA 3

formula

$$\int \frac{dx(P + Rxx)}{\sqrt{(Ax^4 + Cx^2 + E)}}$$

*$Ax^4 + Cx^2 + E$ in factores reales huiusmodi $(f + gxx)(h + kxx)$ resolv
am in aliam transformare, quae per arcus sectionum conicarum cert
queat.*

—

itio princeps: $\theta = 2tu$. Correxit A. K.

Inducatur alia variabilis z, cuius relatio ad x hac ae…

$$4Exxz^4 - 4xxzz\sqrt{AE} = AE \dots + 2\sqrt{AE} \dots$$

ubi $\sqrt{AE}$ erit utique quantitas realis, si quidem $Ax^4 + \dots$ factores binomios reales. Hinc autem fiet

$$\int \frac{dx(P+Rxx)}{\sqrt{(Ax^4+Cx^2+E)}} = \text{Const.} + \dots$$

$$2\int \frac{dz\left(P - \dots\right)}{\sqrt{\left(4E^2z^4 + (C - 6\sqrt{AE})zz + 2A \dots\right)}}$$

in qua nova formula quantitas in denominatore contenta … binomios reales est resolubilis, cum sit

$$(C - 6\sqrt{AE})^2 - 16E\left(2A \dots\right),$$

propterea quod hinc sequitur

$$CC + 4C\sqrt{AE} + 4AE = (C + 2\sqrt{AE})^2 \dots$$

ALITER

Habeat nova variabilis z ad x talem relationem

$$2Exxz^4 - Cxxzz + \frac{CC - 4AE}{8E}xx = 2E \dots$$

eritque

$$\int \frac{dx(P+Rxx)}{\sqrt{(Ax^4+Cxx+E)}} = \frac{CR}{2A\sqrt{E}} + \frac{2E+\dots}{A} \dots$$

$$2\int \frac{dz\left(P - \frac{CR}{2A} + \frac{2ER}{A}zz\right)}{\sqrt{\left(4Ez^4 - 2Czz + \frac{CC - 4AE}{4E}\right)}},$$

cuius denominator pariter certo in factores reales bin…

CONCLUSIO

...monstratis manifestum est hanc formulam

$$\int \frac{dy(\mathfrak{P} + n\mathfrak{B}y + n\mathfrak{A}yy)}{\sqrt{(Ay^4 + 2\mathfrak{B}y^3 + \mathfrak{C}y^2 + 2\mathfrak{D}y + \mathfrak{E})}}$$

... arcus sectionum conicarum construi posse. Cum igitur deno-...mper in duos factores trinomiales reales resolvi possit, haec formula ... potest

$$\int \frac{dy(\mathfrak{P} + n(\alpha\varepsilon + \beta\delta)y + n\alpha\delta yy)}{\sqrt{(\alpha yy + 2\beta y + \gamma)(\delta yy + 2\varepsilon y + \zeta)}},$$

...eadem datur constructio. Porro augendo vel diminuendo y quanti-...nte formula nostra etiam ita repraesentari potest

$$\int \frac{dy(M + Nyy)}{\sqrt{(Ay^4 + Cyy + 2Dy + E)}}.$$

...n fere omnes casus, quos quidem per rectificationem sectionum coni-...grare licet, contineri videntur. Sed in medium afferamus adhuc ...tionem.

PROBLEMA 4

...gare conditiones, sub quibus integrationem huius formulae

$$\int \frac{dy(\mathfrak{P} + \mathfrak{Q}y + \mathfrak{R}yy)}{\sqrt{(\mathfrak{A}y^4 + 2\mathfrak{B}y^3 + \mathfrak{C}yy + 2\mathfrak{D}y + \mathfrak{E})}}$$

...pliciorem

$$\int \frac{dx(P + Qx + Rxx)}{\sqrt{(2Bx^3 + Cx^2 + 2Dx)}}$$

...cat.

SOLUTIO

...tur inter variabiles x et y talis relatio

$$\ldots y + 2xy(\beta x + \gamma y) + \delta xx + \varepsilon yy + 2\zeta xy + 2\eta x + 2\theta y + \varkappa = 0,$$

cuius coefficientes ita determinentur, ut sit

$$\beta\beta - \alpha\delta = 0, \quad \gamma\gamma - \alpha\iota = \mathfrak{A}, \quad \gamma\zeta - \alpha\theta - \beta\iota = \mathfrak{B}, \ldots$$
$$\theta\theta - \iota\varkappa = 0, \quad \eta\eta - \delta\varkappa = \mathfrak{E}, \quad \zeta\eta - \beta\varkappa - \delta\theta = \mathfrak{D} \ldots$$

atque

$$\zeta\zeta + 2\gamma\eta - \alpha\varkappa - \delta\iota - 4\beta\theta = \mathfrak{C},$$

quem in finem definiatur primo p ex hac aequatione cubica

$$p^3 - \frac{1}{2}\mathfrak{C}pp - (\mathfrak{A}\mathfrak{E} - \mathfrak{B}\mathfrak{D})p + \frac{1}{2}(\mathfrak{C}\mathfrak{A}\mathfrak{E} - \mathfrak{A}\mathfrak{D}\mathfrak{D} - \mathfrak{B}\mathfrak{B}\mathfrak{E} \ldots$$

Deinde pro lubitu sumto numero m definiatur q ex hac aequatione

$$qq - q(\mathfrak{D}m - \mathfrak{B}) + (m\mathfrak{C} - p)(mp - \mathfrak{A}) = 0,$$

quo facto, si denuo numerus arbitrarius accipiatur n, erit

$$\beta = \frac{n(m\mathfrak{C} - p)}{\sqrt{(2mp - \mathfrak{A} - mm\mathfrak{C})}}, \qquad \theta = \frac{mp - \mathfrak{A}}{n\sqrt{(2mp - \mathfrak{A} - mm\mathfrak{C})}},$$
$$\alpha = \frac{nq}{\sqrt{(2mp - \mathfrak{A} - mm\mathfrak{C})}}, \qquad \varkappa = \frac{q}{n\sqrt{(2mp - \mathfrak{A} - mm\mathfrak{C})}},$$
$$\delta = \frac{n(m\mathfrak{C} - p)^2}{q\sqrt{(2mp - \mathfrak{A} - mm\mathfrak{C})}}, \qquad \iota = \frac{(mp - \mathfrak{A})^2}{nq\sqrt{(2mp - \mathfrak{A} - mm\mathfrak{C})}},$$
$$\gamma = \frac{m\sqrt{(pp - \mathfrak{A}\mathfrak{C})}}{\sqrt{(2mp - \mathfrak{A} - mm\mathfrak{C})}}, \qquad \eta = \frac{\sqrt{(pp - \mathfrak{A}\mathfrak{C})}}{\sqrt{(2mp - \mathfrak{A} - mm\mathfrak{C})}}$$

et

$$\zeta = \frac{\mathfrak{D}(mp - \mathfrak{A}) - \mathfrak{B}(m\mathfrak{C} - p)}{\sqrt{(pp - \mathfrak{A}\mathfrak{C})(2mp - \mathfrak{A} - mm\mathfrak{C})}}.$$

Quibus inventis erit

$$B = -(\beta\zeta - \alpha\eta - \gamma\delta), \quad D = \zeta\theta - \gamma\varkappa - \iota\eta$$

et

$$C = \zeta\zeta + 2\beta\theta - \alpha\varkappa - \delta\iota - 4\gamma\eta.$$

Ponatur iam

$$\int \frac{dy(\mathfrak{P} + \mathfrak{Q}y + \mathfrak{R}yy)}{\sqrt{(\mathfrak{A}y^4 + 2\mathfrak{B}y^3 + \mathfrak{C}y^2 + 2\mathfrak{D}y + \mathfrak{E})}} = \text{Const.} + mx + ny + \ldots$$
$$\int \frac{dx(P + Qx + Rxx)}{\sqrt{(2Bx^3 + Cx^2 + 2Dx \ldots}}$$

ur ut ante

$$m = \frac{\beta\mathfrak{N}}{\mathfrak{A}}, \quad n = \frac{\gamma\mathfrak{N}}{\mathfrak{A}} \quad \text{et} \quad p = \frac{\alpha\mathfrak{N}}{\mathfrak{A}},$$

$$P = \mathfrak{P} + \frac{(\beta\theta - \gamma\eta)\mathfrak{N}}{\mathfrak{A}}, \quad Q = \frac{B\mathfrak{N}}{\mathfrak{A}} \quad \text{et} \quad R = 0.$$

m est, ut in formula proposita sit $\mathfrak{A}\mathfrak{Q} = \mathfrak{B}\mathfrak{N}$, neque ergo haec os casus suppeditat. At posito $x = zz$ formula transformata abit

$$-2\int \frac{dz(P + Qzz)}{\sqrt{(2Bz^4 + Cz^2 + 2D)}},$$

o saepe facilius succedit quam praecedens.

DE REDUCTIONE FORMULARUM INTEG
AD RECTIFICATIONEM ELLIPSIS AC HY

Commentatio 295 indicis Enestroemiani
Novi commentarii academiae scientiarum Petropolitanae 10 (1764), 1
Summarium ibidem p. 5—9

SUMMARIUM

Quae quantitates numeris neque integris neque fractis neque etiam
nalibus exhiberi possunt, transcendentes vocari solent, quarum ergo va
proxime per numeros exprimere licet. Dari autem huiusmodi quantita
etiamsi ratio ob infinitudinem, quae eas excludere videtur, a plerisque
spiciatur; id quod exemplo notissimo peripheriae circuli, cuius diamet
evidenter declarari potest. Nullum enim est dubium, quin quantita
valorem habeat omnino determinatum, quem adeo primo intuitu consti
4 contineri. Verum intra hos limites innumerabiles constitui possun
denominatorum discrepantes, cuiusmodi simpliciores sunt

$$3\tfrac{1}{2},\quad 3\tfrac{1}{3},\quad 3\tfrac{2}{3},\quad 3\tfrac{1}{4},\quad 3\tfrac{3}{4},\quad 3\tfrac{1}{5},\quad 3\tfrac{2}{5},\quad 3\tfrac{3}{5},\quad 3\tfrac{4}{5},\quad \text{etc.}$$

et generatim in hac forma $3\frac{m}{m+n}$ comprehenduntur; ubi cum tam p
omnes plane numeros substitui liceat, nulla tamen huiusmodi formul
quantitatem praebet, sed quaecunque assumatur, semper a veritate re
continuo minor reddi possit. Deinde quantitates etiam surdas introdu
merorum intra limites 3 et 4 contentorum ulterius in infinitum auge
ab iis, qui in formula $3\frac{m}{m+n}$ continentur, discrepant, neque tamen eti
peritur, qui circuli peripheriam exacte dimetiatur; quamobrem eius q
transcendente habetur. Quod idem multo magis de omnibus circuli
gendum, ita ut, quicunque capiatur sinus in circulo, arcus ipsi responde

sicque solus circulus infinitam quantitatum transcendentium multitudinem le vero etiam logarithmi ad classem numerorum transcendentium sunt ab illis, qui ex circulo nascuntur, prorsus sunt diversi. Iam nemo non fractiones et quantitates transcendentes ex circulo et logarithmis ortae in r, tum inter binos quosvis numeros multitudinem numerorum mediorum mensum augeri; ex quo maxime mirum videbitur ne hoc quidem modo os numeros integros ita numeris mediis expleri, ut iis omnes plane quanti- n terminos contentae exprimi queant. Quin potius praeterea innumera- tum transcendentium genera, tam inter se quam ab illis ex circulo et loga- ime discrepantia, agnosci oportet; inter quae potissimum notari merentur licatione ellipsium et hyperbolarum originem ducunt, propterea quod hae nn sunt notissimae et facillime describuntur. Quomodocunque autem tam yperbola arcus rescindantur, eorum quantitas non solum nullis formulis imi, sed etiam nullo modo neque ad arcus circulares neque ad logarithmos quin etiam singuli arcus tam elliptici quam hyperbolici peculiares quanti- es exhibent, quoniam ne inter se quidem nisi paucissimis casibus exceptis . Ad innumerabilia alia autem quantitatum transcendentium genera calculus , dum omnibus formulis integralibus, quarum integratio algebraice expe- e quantitates transcendentes designantur, in quarum natura evolvenda in- s analystarum maxime coruitur. Cum igitur nunc quidem sit compertum formulas integrales $\int V dx$, si V fuerit functio rationalis ipsius x, semper t arcus circulares exprimi posse, nisi forte algebraicam integrationem ad- integrandi pro iis casibus, quibus V est functio irrationalis ipsius x adhuc rantur, ubi quidem id imprimis esset optandum, ut eae formulae, quibus rrationalis, accuratius evolverentur, quarum integratio per arcus sive ellip- olicos expediri queat. Atque in hac investigatione Auctor istius disserta- st occupatus summumque studium contulit ad hanc formulam integralem plicandam atque adeo ad arcus sive ellipticos sive hyperbolicos reducendam; nulto difficilius est, quam initio videatur. Prout enim quantitatum con- et k aliae fuerint vel positivae vel negativae, casus oriuntur natura sua discrepantes. Primo enim relatio inter has quatuor quantitates ita potest ut formula integralis arcum quendam sive ellipticum sive hyperbolicum nt. Deinde fieri potest, ut integrale binis constet partibus, altera alge- um sive ellipticum sive hyperbolicum exprimente. Praeterea vero etiam casus, quibus integrale neutro modo exhiberi potest, sed praeter partem arcus, alterum ellipticum, alterum hyperbolicum requirit. In tractatione $\int dx \sqrt{\frac{f+gxx}{h+kxx}}$ ob istam varietatem Auctor coactus est duodecim casus con-

stituere, quas singulas operoso calculo ita feliciter expedivit, ut iam fac
quantitates litteris f, g, h, k designentur, integrale concessa ellipsium et
ficatione assignare. Saepenumero autem evenire potest, ut formulae inte
complicatae ope substitutionum idonearum ad talem formam perduci q
omnibus casibus integratio expedita est censenda; ex quo haec investiga
haud leve incrementum attulisse est aestimanda.

Egregia omnino sunt, quae acutissimi Geometrae
D'ALEMBERT[2]) de reductione formularum integralium ad rectific
et Hyperbolae sunt commentati, cum in iis non solum ins
spectatur, sed etiam haud exigua spes affulgeat his rectificatio
aeque commode utendi, atque adhuc arcus circulares et logar
sumus soliti. Nullum enim est dubium, quin haec investi
Geometris tam felici successu suscepta latissime pateat atque
aliquando sit allatura; quamvis enim iam plurima in hoc
stitum, minime tamen totum argumentum quasi exhaustum
Nam postquam longe diversa methodo nova eo perveni, ut tam
Hyperbola diversos arcus definire potuerim, quorum differen
assignare liceat, de quo quidem laudati viri dubitasse videntur
accessio in tractatione huius argumenti expectari poterit. Imp
idoneus signandi modus desiderari videtur, cuius ope arcu
commode in calculo exprimi queant, ac iam logarithmi et ar
insigne Analyseos incrementum per idonea signa in calculum
Talia signa novam quandam calculi speciem suppeditabunt,
prima *elementa exponere* constitui.

Quemadmodum autem omnes arcus circulares ad circul
unitati aequalis statuitur, referri solent, ita etiam pro omn
conicis, quas in calculum recipere volumus, mensuram quand
exprimendum assumi conveniet, quae ad omnes species aeque
spicuum autem est hanc mensuram axi transverso tribui no
in parabola necessario fiat infinitus, in hyperbolis autem neg
consequatur; aeque parum axis coniugatus ad hoc institutum es
quippe qui in parabola quoque fit infinitus et in hyperboli

1) C. MACLAURIN, *A Treatise of fluxions.* Edinburgh 1742, Vol. 2, p.
2) Vide notam 1 p. 235. A. K.

aginarium adipiscitur. Relinquitur igitur parameter, cui quom
tuo valor fixus tribui queat, nihil plane obstat; et quoniam pr
rameter abit in diametrum huiusque semissis unitate exprimi so
uter in sequentibus parametrum *binario* indicabo, ut eius semissi
primatur.

HYPOTHESIS 1

1. *Perpetuo igitur mihi unitas semiparametrum seu semilatus rectum
nicae exprimat.*

COROLLARIUM 1

2. Si ergo a denotet semiaxem transversum, in quo abscissae x
piantur iisque applicatae y normaliter constituantur, habebitur ista

$$yy = 2x - \frac{xx}{a}.$$

COROLLARIUM 2

3. Quamdiu a quantitatem positivam denotat, aequatio erit p
ae quidem, si $a = 1$, abit in circulum; at posito $a = \infty$ habebitur
lores autem negativi ipsius a ad hyperbolas pertinent.

COROLLARIUM 3

4. Ex hac aequatione fit

$$dy = \frac{dx(a - x)}{\sqrt{a(2ax - xx)}}$$

nceque arcus abscissae x respondens

$$= \int \frac{dx\sqrt{(aa - 2a(1-a)x + (1-a)xx)}}{\sqrt{a(2ax - xx)}} \quad \text{seu} \quad = \int dx \sqrt{\Big(\frac{a}{2ax - xx} + $$

ro ellipsi, si fuerit a numerus positivus.

COROLLARIUM 4

5. Posito $a = 1$ fit pro circulo arcus abscissae x, quae est
ersus, respondens $= \int dx \sqrt{\frac{1}{2x - xx}}$, uti constat, ac posito $a = \infty$ pr
olae arcus abscissae x respondens $= \int dx \sqrt{\Big(\frac{1}{2x} + 1\Big)}$.

COROLLARIUM 5

6. Si denique a habeat valorem negativum, puta $a = -c$, erit p
bolis arcus abscissae x respondens $= \int dx \sqrt{\left(\frac{c}{2cx+xx}+\frac{c+1}{c}\right)}$.

HYPOTHESIS 2

7. *In sectione conica, cuius semiparameter $= 1$ et semiaxis transve
atque abscissae in axe transverso a vertice capiantur, arcum abscissae
dentem hac scriptione $\Pi x\,[a]$ indicabo.*

COROLLARIUM 1

8. Post signum ergo Π scribetur abscissa in axe transverso
computata, cui adiungetur semiaxis transversus intra uncinulas [ ] i

COROLLARIUM 2

9. Haec ergo expressio $\Pi x\,[a]$ designat arcum ellipticum, si a
titas positiva, et circularem quidem, si $a = 1$, cuius sinus versus
si $a = \infty$, exprimit ea arcum parabolicum, ac denique si a sit quan
gativa, arcum hyperbolicum.

COROLLARIUM 3

10. Habet ergo huiusmodi expressio $\Pi x\,[a]$ valorem determinata
non solum sectio conica definitur, sed etiam eius arcus illa ex
indicatur.

COROLLARIUM 4

11. Manifestum autem est, ut istius expressionis valor fiat r
scissam x non solum realem, sed etiam positivam esse debere.
praeterea, si fuerit a quantitas positiva, necesse est, ut abscissa x
$2a$ non transgrediatur. Quantitatem a autem necessario realem esse

COROLLARIUM 5

12. Haec ergo expressio $\Pi x\,[a]$ imaginaria erit, si vel 1° n
fuerit imaginarius, vel 2° x quantitas imaginaria, vel 3° quantitas
vel 4° positiva quidem, sed maior quam $2a$, si scilicet a sit quantitas

COROLLARIUM 6

13. Notetur quoque hanc formulam $\Pi x[a]$ eiusmodi functionem
xhibere, quae evanescat evanescente x, ita ut sit $\Pi 0[a]=0$. Sin au
quantitas infinite parva $=\omega$, erit $\Pi\omega[a]=\sqrt{2\omega}$ neque ergo ab a

THEOREMA 1

14. *Si haec formula differentialis $dx\sqrt{\left(\frac{a}{2ax-xx}+\frac{a-1}{a}\right)}$ ita integr*
tegrale evanescat posito $x=0$, erit

$$\int dx\sqrt{\left(\frac{a}{2ax-xx}+\frac{a-1}{a}\right)}=\Pi x[a].$$

DEMONSTRATIO

Utraque enim expressio refertur ad sectionem conicam, cuius se
eter $=1$ et semiaxis transversus $=a$, atque arcum eius denotat a
ntum, qui abscissae x respondet abscissa in axe transverso su
ariter a vertice computata.

COROLLARIUM 1

15. Si pro a scribamus $-a$, habebitur

$$\int dx\sqrt{\left(\frac{a}{2ax+xx}+\frac{a+1}{a}\right)}=\Pi x[-a],$$

uo casu, si quantitas uncinulis inclusa sit negativa, arcus hype
dicatur.

COROLLARIUM 2

16. Si sit $a=\infty$, quo casu prodit rectificatio parabolae, erit

$$\int dx\sqrt{\left(\frac{1}{2x}+1\right)}=\Pi x[\infty],$$

ius valor, uti constat, per logarithmos exhiberi potest.

COROLLARIUM 3

17. At si sit $a = 1$, ut habeatur

$$\int \frac{dx}{\sqrt{(2x - xx)}} = \Pi x[1],$$

hac expressione arcus circuli, cuius radius $= 1$, exprimitur, cui
$= x$; eius ergo cosinus erit $= 1 - x$ et sinus $= \sqrt{(2x - xx)}$.

COROLLARIUM 4

18. Cum eidem abscissae x geminus arcus, alter positivus,
respondeat, expressio $\Pi x[a]$ per se geminum exhibebit valore
signa radicalia quadratica; erit ergo functio biformis, tam vale
quam positivum continens.

SCHOLION

19. Quoties autem expressio $\Pi x[a]$ ad ellipsin refertur,
duos, verum adeo infinitos valores complectitur, perinde uti in
dantur arcus eidem sinui verso x convenientes. Naturam erg
onis infinitiformis pro ellipsibus accuratius perpendamus.

PROBLEMA 1

20. *Invenire omnes arcus ellipticos eidem abscissae x respo*
finire omnes valores formulae $\Pi x[a]$ convenientes.

SOLUTIO

Sit z minimus arcus abscissae x respondens in ellipsi,
transversus est $= a$; ponatur semiperimeter ellipsis $= A$, ut sit
$= 2A$, atque manifestum est eidem abscissae x etiam res
$2A - z$, $2A + z$, $4A - z$, $4A + z$, $6A - z$, $6A + z$ etc., qui o
negativis continentur in formula $\Pi x[a]$, ita ut eius valor
$\pm 2nA \pm z$ denotante n numerum integrum quemcunque.

COROLLARIUM 1

. Cum $\frac{1}{2}A$ sit quarta pars perimetri ellipsis eique abscissa $x = a$ co
orit $\frac{1}{2}A = \Pi a[a]$, semiperimetro autem A convenit abscissa $2a$, un
$2a[a]$, ergo $\Pi 2a[a] = 2\Pi a[a]$.

COROLLARIUM 2

. Si capiatur abscissa $= 2a - x$, erit arcus ei respondens $= A - \Pi x[$
olligitur haec aequalitas

$$\Pi x[a] + \Pi(2a - x)[a] = 2\Pi a[a],$$

cula ( ), quibus abscissa inscribitur, ab uncinulis [ ] semiaxem tran
continentibus probe distingui oportet.

COROLLARIUM 3

. Eadem aequalitas ex integrali potest colligi; posito enim $2a -$
erit

$$\Pi(2a - x)[a] = -\int dx \sqrt{\left(\frac{a}{2ax - xx} + \frac{a - \frac{1}{a}}{a}\right)} = -\Pi x[a] + \text{Const.}$$

s vero ex quodam casu debet colligi. Scilicet si ponatur $x = 0$, f
$= \Pi 2a[a]$; vel si ponatur $x = a$, prodit

$$\text{Const.} = \Pi a[a] + \Pi a[a] = 2\Pi a[a].$$

SCHOLION

. Arcus elliptici praeterea hanc habent proprietatem, ut, si axis tran
$2a$ minor fuerit parametro, quod scilicet evenit, si axis minor p
rso capiatur, iidem arcus sumi possint in alia ellipsi, cuius axis s
parametro. Nititur haec reductio similitudine ellipsium, quarum sem
nt a et $\frac{1}{a}$ manente parametro eadem $= 2$.

PROBLEMA 2

25. *Arcum ellipticum $\Pi x[a]$, si fuerit $a < 1$, ad alium et cuius semiaxis sit unitate maior.*

SOLUTIO

Cum sit

$$\Pi x[a] = \int dx \sqrt{\left(\frac{a}{2ax - xx} + 1 - \frac{1}{a}\right)},$$

statuatur

$$\sqrt{(2ax - xx)} = a - aay$$

eritque

$$2ax - xx = aa(1 - 2ay + aayy) \quad \text{hincque} \quad a - x = a\sqrt{(2a$$

unde fit

$$dx = \frac{-aady(1 - ay)}{\sqrt{(2ay - aayy)}}.$$

Facta hac substitutione consequemur

$$\Pi x[a] = \int \frac{-aady(1-ay)}{\sqrt{(2ay - aayy)}} \sqrt{\left(\frac{1}{a(1-ay)^2} + 1 - \frac{1}{a}\right)}$$

seu

$$\Pi x[a] = -a\sqrt{a} \cdot \int dy \sqrt{\frac{1 + (a-1)(1-ay)^2}{2ay - aayy}},$$

quae expressio reducitur ad hanc formam

$$\Pi x[a] = -a\sqrt{a} \cdot \int dy \sqrt{\left(\frac{1}{2y - ayy} + 1 - a\right)}.$$

Ponamus in formula integrali $a = \frac{1}{b}$, ut sit $b = \frac{1}{a}$, ac fiet ea

$$\int dy \sqrt{\left(\frac{b}{2by - yy} + 1 - \frac{1}{b}\right)} = \Pi y[b] + \text{Const.}$$

Quare restituta littera a obtinebitur ob $y = \frac{a - \sqrt{(2ax - xx)}}{aa}$

$$\Pi x[a] = \text{Const.} - a\sqrt{a} \cdot \Pi \frac{a - \sqrt{(2ax - xx)}}{aa}\left[\frac{1}{a}\right],$$

n $x=0$ definitur constans $= a\sqrt{a}\cdot \Pi\frac{1}{a}\left[\frac{1}{a}\right]$, ita ut sit

$$\Pi x[a] = a\sqrt{a}\cdot \Pi\frac{1}{a}\left[\frac{1}{a}\right] - a\sqrt{a}\cdot \Pi\frac{a-\sqrt{(2ax-xx)}}{aa}\left[\frac{1}{a}\right],$$

s ellipsis, cuius semiaxis est a, reductus est ad arcus alius ellipsis, axis est $=\frac{1}{a}$.

COROLLARIUM 1

ponatur $x=a$, fit

$$\frac{a-\sqrt{(2ax-xx)}}{aa} = 0 \quad \text{ideoque} \quad \Pi a[a] = a\sqrt{a}\cdot \Pi\frac{1}{a}\left[\frac{1}{a}\right].$$

rimeter prioris ellipsis, cuius semiaxis $=a$, est ad perimetrum cuius semiaxis $=\frac{1}{a}$, uti $a\sqrt{a}$ ad 1 seu ut $a^{\frac{3}{4}}$ ad $\frac{1}{a^{\frac{3}{4}}}$.

COROLLARIUM 2

um arcus abscissae $\frac{a-\sqrt{(2ax-xx)}}{aa}$ respondens posito $x=0$ fiat , hinc aucto x decrescat, donec evanescat posito $x=a$, lex conxigit, ut sumto $x>a$ iste arcus negativum obtineat valorem.

COROLLARIUM 3

ucto ergo $x=2a$ erit

$$\Pi\frac{a-\sqrt{(2ax-xx)}}{aa}\left[\frac{1}{a}\right] = -\Pi\frac{1}{a}\left[\frac{1}{a}\right],$$

fiet hoc casu $x=2a$

$$\Pi 2a[a] = 2\,\Pi a[a] = 2a\sqrt{a}\,\Pi\frac{1}{a}\left[\frac{1}{a}\right],$$

nsentit cum coroll. 1.

SCHOLION

ubstitutione hic adhibita $\sqrt{(2ax-xx)} = a - aay$ formulam intealiam sui similem transmutavimus, cuius valor per arcum alius hiberi poterat. Si autem aliis substitutionibus utamur, semper

adipiscimur formulas integrales, quarum integratio per ro
onum conicarum expediri potest; quia vero a tam negativu
valorem recipere potest, substitutiones eaedem tam ad ell
bolas extendi possunt.

PROBLEMA 3

30. *Formulam integralem*

$$\int dx \sqrt{\left(\frac{a}{2ax-xx}+1-\frac{1}{a}\right)}$$

*per substitutiones idoneas in alias formulas concinniores transfor
semper futurus sit* $= \Pi x[a]$.

SOLUTIO

Prima reductio fit ponendo $x=a-naz$, quo facto
induit hanc formam

$$\int -ndz\sqrt{\frac{aa-nna(a-1)zz}{1-nnzz}} = \Pi a(1-nz)[$$

multiplicetur ea per m, ut sit

$$\int -dz\sqrt{\frac{m^2n^2aa+m^2n^4a(a-1)zz}{1-nnzz}} = m\Pi a(1-n$$

quam expressionem iam ad hanc formam, concinnam ae
reducere licet

$$\int dz\sqrt{\frac{f+gzz}{h+kzz}};$$

fieri scilicet oportet

$$m^2n^2a^2h=f, \quad m^2n^4a(1-a)h=g, \quad -nnh=$$

unde ob $nnh=-k$ et $n=\sqrt{\frac{-k}{h}}$ erit

$$-mmaak=f, \quad mma(1-a)kk=gh$$

hincque

$$\frac{(a-1)k}{a}=\frac{gh}{f} \quad \text{et} \quad a=\frac{fk}{fk-gh}.$$

$$m = \frac{1}{a}\sqrt{\frac{-f}{k}} \quad \text{seu} \quad m = \frac{fk-gh}{fk}\sqrt{\frac{-f}{k}},$$

loribus concluditur fore

$$\frac{+gzz}{+kzz} = C - \frac{fk-gh}{fk}\sqrt{\frac{-f}{k}}\, \Pi\, \frac{fk}{fk-gh}\left(1 - z\sqrt{\frac{-k}{h}}\right)\left[\frac{fk}{fk-gh}\right].$$

egrale, nisi forma sit imaginaria, per rectificationem ellipsis ab-
ierit $\frac{fk}{fk-gh}$ quantitas positiva; sin autem sit negativa, integratio
bolicum indicat.

COROLLARIUM 1

ergo haec forma ab imaginariis sit libera, necesse est, ut tam
sit quantitas positiva. Si alterutra vel ambae fuerint negativae,
ginariis implicatur; nihilo vero minus eius valor erit realis, si
ntiale ipsum sit reale.

COROLLARIUM 2

autem formula differentialis ponatur realis, assumere licet tam
n $h+kzz$ esse quantitates positivas; si enim ambae essent
tatis signis ad positivas reduci possent. Ita statuamus esse

$$f + gzz > 0 \quad \text{et} \quad h + kzz > 0.$$

COROLLARIUM 3

utem formula nostra inventa arcum realem sectionis conicae ex-
sufficit esse $\sqrt{\frac{-f}{k}}$ et $\sqrt{\frac{-k}{h}}$ quantitates reales, sed praeterea re-
abscissa sit positiva; ubi duos casus perpendi convenit, prout
fuerit ellipsis vel hyperbola.

COROLLARIUM 4

ergo primo sectio conica ellipsis seu $\frac{fk}{fk-gh}$ quantitas positiva
o est, ut sit $1 - z\sqrt{\frac{-k}{h}} > 0$ seu $1 > \frac{-kzz}{h}$, unde fit $\frac{h+kzz}{h} > 0$.
thesin est $h+kzz > 0$. Quare casu, quo $\frac{fk}{fk-gh} > 0$, ad reali-
r requiritur, ut h sit quantitas positiva.

34*

COROLLARIUM 5

35. Pro hyperbola, seu si $\frac{fk}{fk-gh}$ fuerit quantitas
nostra ita debet repraesentari

$$\int dz\sqrt{\frac{f+gzz}{h+kzz}} = C + \frac{gh-fk}{fk}\sqrt{\frac{-f}{k}}\,\Pi_{\frac{fk}{gh-fk}}\Big(z\sqrt{\frac{-}{h}}$$

ita ut $\frac{fk}{gh-fk}$ iam sit quantitas positiva. Necesse autem

$$z\sqrt{\frac{-k}{h}} > 1 \quad \text{seu} \quad \frac{h+kzz}{h} < 0,$$

quare ob $h+kzz>0$ arcus hyperbolicus non erit real
titas negativa.

COROLLARIUM 6

36. Pro ellipsi ergo, seu si sit $\frac{fk}{fk-gh}>0$, nostra ex
nebit realem, si fuerit

$$1.\ h>0, \quad 2.\ k<0 \quad \text{ac} \quad 3.\ f>0.$$

Pro hyperbola autem, seu si $\frac{fk}{gh-fk}>0$, arcus erit realis

$$1.\ h<0, \quad 2.\ k>0 \quad \text{et} \quad 3.\ f<0.$$

SCHOLION 1

37. Ope formulae igitur inventae nonnisi aliquot casu

$$\int dz\sqrt{\frac{f+gzz}{h+kzz}}$$

expedire possumus. Nempe cum in genere litterae f, g
sive positivas sive negativas significent, si iam ad cas
tantum pro positivis assumamus, sequentes integrationes

$$\text{I.}\ \int dz\sqrt{\frac{f+gzz}{h-kzz}} = C - \frac{fk+gh}{fk}\sqrt{\frac{f}{k}}\,\Pi_{\frac{fk}{fk+gh}}\Big(1-$$

$$\text{II.}\ \int dz\sqrt{\frac{f-gzz}{h-kzz}} = C - \frac{fk-gh}{fk}\sqrt{\frac{f}{k}}\,\Pi_{\frac{fk}{fk-gh}}\Big(1-$$

at hoc casu requiritur, ut sit $fk>g$

$$\sqrt{\frac{-f+gzz}{-h+kzz}} = C + \frac{gh-fk}{fk}\sqrt{\frac{f}{k}}\,\Pi\,\frac{fk}{gh-fk}\left(z\sqrt{\frac{k}{h}}-1\right)\left[\frac{-fk}{gh-fk}\right];$$

hoc vero casu requiritur, ut sit $gh > fk$.

n hoc tertio casu indoles litterarum f, h, k iam sit definita, pro g itatem negativam assumere non liceat, hos tantum tres casus per blema expedire licet. Reliqui vero omnes excluduntur, dum ad narios perducuntur. Interim tamen cum certo habeant valores nadmodum hi per alios arcus reales exprimi queant, in sequentiabimus.

SCHOLION 2

equam autem hoc opus suscipiamus, e re erit omnes casus pro ignorum, quibus litterae f, g, h, k affectae esse possunt, enumeiam fieri potest, ut quidam ob aliam conditionem in binos subnt, quemadmodum supra in secundo et tertio usu venit. Hac liecta sequentes 12 habebimus casus, ubi quidem litterae f, g, h, k ivos valores habere accipiuntur.

I. $\int dz\sqrt{\frac{f+gzz}{h+kzz}}$, si fuerit $fk > gh$.

II. $\int dz\sqrt{\frac{f+gzz}{h+kzz}}$, si fuerit $gh > fk$.

III. $\int dz\sqrt{\frac{f+gzz}{h-kzz}}$ nulla limitatione adiuncta.

IV. $\int dz\sqrt{\frac{f+gzz}{kzz-h}}$ nulla limitatione adiuncta.

V. $\int dz\sqrt{\frac{f-gzz}{h+kzz}}$ nulla limitatione adiuncta.

VI. $\int dz\sqrt{\frac{f-gzz}{h-kzz}}$, si fuerit $fk > gh$.

VII. $\int dz\sqrt{\frac{f-gzz}{h-kzz}}$, si fuerit $fk < gh$.

VIII. $\int dz\sqrt{\frac{f-gzz}{-h+kzz}}$; hic necessario est $fk > gh$.

IX. $\int dz\sqrt{\frac{-f+gzz}{h+kzz}}$ nulla limitatione adiuncta.

$$\text{X. } \int dz \sqrt{\frac{-f+gzz}{h-kzz}}; \text{ hic necessario est } f$$

$$\text{XI. } \int dz \sqrt{\frac{-f+gzz}{-h+kzz}}, \text{ si fuerit } fk > gh.$$

$$\text{XII. } \int dz \sqrt{\frac{-f+gzz}{-h+kzz}}, \text{ si fuerit } fk < gh.$$

Atque ex his duodecim casibus hactenus tantum tres, s
conficere licuit, quorum integralia per arcus simplices
exprimuntur.

SCHOLION 3

39. Quanquam autem his tribus casibus integrali
ticos sive hyperbolicos expressimus, tamen quaedam
litteras f, g, h et k, quibus nostra expressio tantis inco
verus valor integralis inde erui nequeat, etiamsi per
Ac primo quidem in genere, si in formula

$$\int dz \sqrt{\frac{f+gzz}{h+kzz}}$$

fuerit $fk = gh$, valor integralis ita quantitatibus eva
involvitur, ut eius vera quantitas inde perspici nequea
se sit planissima; posito enim $k = \frac{gh}{f}$ erit

$$\int dz \sqrt{\frac{f+gzz}{h+kzz}} = \int dz \sqrt{\frac{f(f+gzz)}{h(f+gzz)}} = \int dz \sqrt{\frac{f}{h}}$$

ita ut revera sit ob $gh = fk$

$$C - \frac{fk-fk}{fk}\sqrt{\frac{-f}{k}}\, \Pi\frac{fk}{fk-fk}\left(1 - z\sqrt{\frac{-k}{h}}\right)\Big[\frac{}{fk}$$

etsi ratio huius aequalitatis difficulter perspiciatur, cu
arcum parabolicum abscissae infinitae respondentem, q
evanescentem sit multiplicatus, indicare videatur. Int
damus in parabola arcum, qui abscissae infinitae res
rationem aequalitatis habere, erit

$$\Pi\frac{fk}{fk-fk}\left(1 - z\sqrt{\frac{-k}{h}}\right)\left[\frac{fk}{fk-fk}\right] = \frac{fk}{fk-fk}\Big(1$$

rcus per factorem

$$-\frac{(fk-fk)}{fk}\sqrt{\frac{-f}{k}}$$

plicatus praebet productum finitum

$$= -\left(1 - z\sqrt{\frac{-k}{h}}\right)\sqrt{\frac{-f}{k}} = -\sqrt{\frac{-f}{k}} + z\sqrt{\frac{f}{h}},$$

valor cum veritate egregie congruit. Reliquas difficultates casus pa
es percurrentes seorsim examinemus.

INTEGRATIO CASUS III

$$\int dz\sqrt{\frac{f+gzz}{h-kzz}} = C - \frac{fk+gh}{fk}\sqrt{\frac{f}{k}}\,\Pi_{\frac{fk}{fk+gh}}\left(1 - z\sqrt{\frac{k}{h}}\right)\left[\frac{fk}{fk+gh}\right]$$

10. Si f, g, h, k denotent quantitates nihilo maiores, arcus ellipti
ntegrali facile assignatur; *neque turbat casus, quo* $g=0$, quippe
rcum circularem expeditur eritque

$$\int\frac{dz\sqrt{f}}{\sqrt{(h-kzz)}} = C - \sqrt{\frac{f}{k}}\,\Pi\left(1 - z\sqrt{\frac{k}{h}}\right)[1].$$

e h evanescere nequit, quin simul formula differentialis ipsa fiat i
ia. At si f vel k evanescat, quorum priori casu integrale est algeb
posteriori vero per logarithmos dari potest, nostra formula refertur
in evanescentem nihilque inde concludere licet; mox autem pro eod
aliam integralis formam exhibebimus, unde vera integralis quanti
as elici poterit.

INTEGRATIO CASUS VI

$$\int dz\sqrt{\frac{f-gzz}{h-kzz}} = C - \frac{fk-gh}{fk}\sqrt{\frac{f}{k}}\,\Pi_{\frac{fk}{fk-gh}}\left(1 - z\sqrt{\frac{k}{h}}\right)\left[\frac{fk}{fk-gh}\right]$$

SI FUERIT $fk > gh$

41. Hic iterum nulla difficultas occurrit, quicunque valores litteris f
k tribuantur, dummodo sit $fk > gh$; semper enim integrale per arc
icum exprimitur neque etiam negotium facessit casus $g=0$, quo
arcus circularis denotatur. At si sit $k=0$, neque enim f et h in n

lum abire possunt, conditio $fk > gh$ non amplius sal
nullum incommodum locum habet, praeter id, quo est
iam ante in genere expedivimus.

INTEGRATIO CASUS XII

$$\int dz \sqrt{\frac{-f+gzz}{-h+kzz}} = C + \frac{gh-fk}{fk}\sqrt{\frac{f}{k}}\,\Pi_{gh-fk}^{fk}\Big(z\sqrt{}$$

SI FUERIT $gh > fk$

42. Hoc casu integrale arcu hyperbolico definitur,
potest fieri negativum. Si fuerit $f=0$, quo casu int
axis hyperbolae evanescit neque hinc valor integral
$h=0$, conditio necessaria $gh > fk$ evertitur, difficultas e
subsistit, quae autem in aliis formulis infra pro eod
tolletur.

PROBLEMA 4

43. *Formulam integralem*

$$\int dz \sqrt{\frac{f+gzz}{h+kzz}}$$

per substitutionem in aliam sui similem transformare.

SOLUTIO

Tentanti huiusmodi substitutionem $z = \sqrt{\frac{\alpha+\beta xx}{\gamma+\delta xx}}$
$x = \sqrt{(h+kzz)}$; unde fit

$$z = \sqrt{\frac{xx-h}{k}}, \quad dz = \frac{xdx}{\sqrt{k(xx-h)}} \quad \text{et} \quad f+gzz=$$

ideoque

$$\int dz \sqrt{\frac{f+gzz}{h+kzz}} = \frac{1}{k}\int dx \sqrt{\frac{fk-gh+g}{xx-h}}$$

quae locum habet, quoties k est quantitas positiva, quo
est quantitas positiva. Sin autem k fuerit quantitas n
positiva, transformatio ita est repraesentanda

$$\int dz \sqrt{\frac{f+gzz}{h+kzz}} = \frac{1}{k}\int dx \sqrt{\frac{gh-fk-g}{h-xx}}$$

COROLLARIUM 1

mparantes formulam

$$\int dx\sqrt{\frac{fk-gh+gxx}{xx-h}}$$

a initio generatim integrata

$$\int dz\sqrt{\frac{f+gzz}{h+kzz}}$$

$z=x$, $f=fk-gh$, $g=g$, $h=-h$, $k=1$, unde $fk-gh=fk$ et

$$\frac{gh+gxx}{x-h}=C-\frac{fk}{fk-gh}\sqrt{(-fk+gh)}\,\Pi\,\frac{fk-gh}{fk}\left(1-x\sqrt{\frac{1}{h}}\right)\left[\frac{fk-gh}{fk}\right].$$

COROLLARIUM 2

stituto hoc valore, cum sit $x=\sqrt{(h+kzz)}$, erit

$$\frac{f+gzz}{h+kzz}=C+\frac{f}{\sqrt{(gh-fk)}}\,\Pi\,\frac{gh-fk}{fk}\left(\frac{\sqrt{(h+kzz)}}{\sqrt{h}}-1\right)\left[\frac{-gh+fk}{fk}\right],$$

ut sit realis, necesse est, ut primo sit $gh-fk>0$, tum vero
. Erit ergo ad hyperbolam, si fk sit quantitas positiva et k po-
llipsin esse nequit, nisi sit k quantitas negativa, f vero positiva,
su esse debet fk quantitas negativa.

COROLLARIUM 3

mili modo altera forma $\int dx\sqrt{\frac{gh-fk-gxx}{h-xx}}$ cum canonica
$\frac{z}{z}$ comparata dat $z=x$, $f=gh-fk$, $g=-g$, $h=h$ et $k=-1$;
$h=fk$ et

$$\frac{fk-gxx}{-xx}=C-\frac{fk}{fk-gh}\sqrt{(gh-fk)}\,\Pi\,\frac{fk-gh}{fk}\left(1-x\sqrt{\frac{1}{h}}\right)\left[\frac{fk-gh}{fk}\right].$$

COROLLARIUM 4

bstituto ergo pro x valore $\sqrt{(h+kzz)}$ erit ut ante

$$\sqrt{\frac{f+gzz}{h+kzz}}=C+\frac{f}{\sqrt{(gh-fk)}}\,\Pi\,\frac{gh-fk}{-fk}\left(1-\frac{\sqrt{(h+kzz)}}{\sqrt{h}}\right)\left[\frac{gh-fk}{-fk}\right],$$

quae locum habere nequit, nisi $gh - fk$ et h sit qu
ellipsi erit, si k sit quantitas negativa et f positiva
hyperbola, si k et g sint positivae, quemadmodum iam
ut hos duos casus distinguere non opus fuerit.

COROLLARIUM 5

48. Geminis his integralibus formulae generalis
collatis habebimus

$$\Pi_{\frac{fk}{fk-gh}}\left(1 - z\sqrt{\frac{-k}{h}}\right)\left[\frac{fk}{fk-gh}\right] = \frac{-fk\sqrt{fk}}{(fk-gh)^{\frac{3}{2}}}\Pi^{\frac{fk-gh}{fk}}\left(1 \right.$$

quae aequalitas posito ad abbreviandum

$$\frac{fk}{fk-gh} = \frac{m}{n} \quad \text{et} \quad z\sqrt{\frac{-k}{h}} = t$$

abit in hanc formam

$$\Pi^{m}_{n}(1-t)\left[\frac{m}{n}\right] = \frac{m\sqrt{m}}{n\sqrt{n}}\Pi^{n}_{m}\left(1 - \sqrt{(1 - \right.$$

COROLLARIUM 6

49. Arcus igitur ellipticus quicunque responde
semiaxe existente $= \frac{m}{n}$ reducitur ad arcum alius ellip
$= \frac{n}{m}$ et abscissa $= 1 - \sqrt{(1 - tt)}$, hunc arcum per $\frac{m\sqrt{}}{n\sqrt{}}$
aequalitatis ratio est similitudo harum duarum ellipsiu
arcus hyperbolicus ad alium reduci nequit, quia ob
imaginarium.

SCHOLION

50. Hinc novas integrationes nanciscimur realit
suggerit § 45 arcum hyperbolicum involventem, ubi
runtur

1. $h > 0$, 2. $k > 0$, 3. $f > 0$ et 4.

unde ob $h > 0$ erit quoque $g > 0$; hisque casus II § 3
...tur. Deinde arcus ellipticus negotium conficiet his

$k < 0$, 2. $h > 0$, 3. $gh - fk > 0$ et 4. $f >$

n nunc positive et negative capi potest. Si sumatur positive, III, sin negative, casus VI, qui quidem iam supra sunt soluti. enere notandum omnes arcus ellipticos duplici modo exprimi ragraphum praecedentem. Integralia ergo horum trium casuum unt.

INTEGRATIO CASUS II

$$\ldots + gzz \over \ldots + kzz = C + \frac{f}{\sqrt{(gh-fk)}} \Pi \frac{gh-fk}{fk} \left(\frac{\sqrt{(h+kzz)}}{\sqrt{h}} - 1 \right) \left[\frac{-gh+fk}{fk} \right]$$

SI FUERIT $gh > fk$

conditionem $gh > fk$ neque g neque h evanescere potest. Si f yperbola abit in parabolam, cuius arcus abscissae infinitae re- indicatur, qui ergo abscissae aequalis est censendus; unde pro abebitur istud integrale

$$\ldots \sqrt{\frac{gzz}{h+kzz}} = C + \frac{\sqrt{gh}}{k} \left(\frac{\sqrt{(h+kzz)}}{\sqrt{h}} - 1 \right) = C + \frac{\sqrt{g(h+kzz)}}{k},$$

omnino est consentaneum.

SCHOLION

er casus moram facessens est, quo $k=0$ et hyperbola iterum bolam. At ob $k=0$ erit

$$\sqrt{\left(1 + \frac{kzz}{h}\right)} - 1 = \frac{kzz}{2h};$$

tio per arcum parabolicum absolvetur hoc modo

$$\int dz \sqrt{\frac{f \pm gzz}{h}} = C + \frac{f}{\sqrt{gh}} \Pi \frac{gzz}{2f} [\infty],$$

operatione consueta elicitur. Si insuper esset $g=0$, ob

$$\Pi \frac{gzz}{2f} = z\sqrt{\frac{g}{f}}$$

$$\int dz \sqrt{\frac{f}{h}} = C + z\sqrt{\frac{f}{h}}.$$

$$\int dz\sqrt{\frac{f+gzz}{h-kzz}}=C+\frac{f}{\sqrt{(gh+fk)}}\,\Pi\,\frac{gh+fk}{fk}\Big(1-$$

SINE ULLA LIMITATIONE

53. Hinc casus $h=0$ sponte excluditur; unde
quuntur. Si primo sit $k=0$, ellipsis abit in parabol

$$1-\sqrt{\Big(1-\frac{kzz}{h}\Big)}=\frac{kzz}{2h}$$

habetur ut ante

$$\int dz\sqrt{\frac{f+gzz}{h}}=C+\frac{f}{\sqrt{gh}}\,\Pi\,\frac{gzz}{2f}$$

si deinde sit $f=0$, denuo parabola et arcus absci
ideoque aequalis censendus prodit, unde fit

$$\int dz\sqrt{\frac{gzz}{h-kzz}}=C+\frac{\sqrt{gh}}{k}\Big(1-\frac{\sqrt{(h-kzz)}}{\sqrt{h}}\Big)=$$

si tertio sit $g=0$, ellipsis abit in circulum fitque

$$\int dz\sqrt{\frac{f}{h-kzz}}=C+\frac{f}{\sqrt{fk}}\,\Pi\Big(1-\frac{\sqrt{(h}}{}$$

sicque casus difficiliores supra § 40 memoratos hic

INTEGRATIO CASUS VI

$$\int dz\sqrt{\frac{f-gzz}{h-kzz}}=C+\frac{f}{\sqrt{(fk-gh)}}\,\Pi\,\frac{fk-gh}{fk}\Big(1-$$

SI MODO FUERIT $fk>gh$

54. Hoc casu aeque ac supra § 41, ubi idem
difficultas relinquitur, quia ob $fk>gh$ neque f n
neque vero etiam h in nihilum abire potest, quin
negativus. At si $g=0$, nulla occurrit difficultas,
circularem revocetur.

SCHOLION

ergo reductione id sumus lucrati, ut iam praeter casus III, VI evolutos etiam casum II expediverimus. Reliqui vero octo modo per arcus simplices reales integrari possunt, sed praeterea braicam continent; quin etiam nonnulli praeter hanc partem algeos arcus, alterum ellipticum, alterum hyperbolicum, complectuntur. em integralia investiganda necesse est, ut alias formulas integrales er variabilem z bina radicalia $\sqrt{(f+gzz)}$ et $\sqrt{(h+kzz)}$ involmplemur, quae ad formam $\int dx\sqrt{\frac{\alpha+\beta xx}{\gamma+\delta xx}}$ sint reductibiles.

PROBLEMA 5

s formulas integrales praeter z bina radicalia

$$\sqrt{(f+gzz)} \quad et \quad \sqrt{(h+kzz)}$$

venire, quarum integratio ad formam

$$\int dx\sqrt{\frac{\alpha+\beta xx}{\gamma+\delta xx}}$$

SOLUTIO

substitutionibus, quarum praecipuas hic percurramus.

$=\frac{1}{z}$; erit $z=\frac{1}{x}$, $\sqrt{(f+gzz)}=\frac{\sqrt{(fxx+g)}}{x}$ et $\sqrt{(h+kzz)}=\frac{\sqrt{(hxx+k)}}{x}$.
$=-\frac{dz}{zz}$ erit

$$dx\sqrt{\frac{fxx+g}{hxx+k}}=-\frac{dz}{zz}\sqrt{\frac{f+gzz}{h+kzz}}$$

$$\int\frac{dz}{zz}\sqrt{\frac{f+gzz}{h+kzz}}=-\int dx\sqrt{\frac{fxx+g}{hxx+k}}.$$

tur $x=\sqrt{(f+gzz)}$; erit $dx=\frac{gzdz}{\sqrt{(f+gzz)}}$, $z=\sqrt{\frac{xx-f}{g}}$ et $\sqrt{\frac{gh-fk+kxx}{g}}$, unde conficitur

$$dx\sqrt{\frac{xx-f}{gh-fk+kxx}} = \frac{gzzdz}{\sqrt{(f+gzz)(h+}}$$

et

$$dx\sqrt{\frac{gh-fk+kxx}{xx-f}} = gdz\sqrt{\frac{h+kzz}{f+gzz}} \quad \text{(o}$$

quare erit

$$\int\frac{zzdz}{\sqrt{(f+gzz)(h+kzz)}} = \frac{1}{g}\int dx\sqrt{\frac{xx}{gh-f}}$$

3. Si ponatur $x = \sqrt{(h+kzz)}$, erit simili modo

$$\int\frac{zzdz}{\sqrt{(f+gzz)(h+kzz)}} = \frac{1}{k}\int dx\sqrt{\frac{xx}{fk-g}}$$

4. Sit $x = \frac{1}{\sqrt{(f+gzz)}}$; erit $dx = \frac{-gzdz}{(f+gzz)^{\frac{3}{2}}}$, $z = \sqrt{(}$

et $\sqrt{(h+kzz)} = \sqrt{\frac{k+(gh-fk)xx}{gxx}}$, unde fit

$$dx\sqrt{\frac{1-fxx}{k+(gh-fk)xx}} = \frac{-gzzdz}{(f+gzz)^{\frac{3}{2}}\sqrt{(h}}$$

et

$$dx\sqrt{\frac{k+(gh-fk)xx}{1-fxx}} = \frac{-gdz\sqrt{(h+}}{(f+gzz)}$$

hincque

$$\int\frac{zzdz}{(f+gzz)^{\frac{3}{2}}\sqrt{(h+kzz)}} = -\frac{1}{g}\int dx\sqrt{\frac{}{k+(}}$$

et

$$\int\frac{dz\sqrt{(h+kzz)}}{(f+gzz)^{\frac{3}{2}}} = -\frac{1}{g}\int dx\sqrt{\frac{k+(gh}{1-}}$$

5. Simili modo si ponatur $x = \frac{1}{\sqrt{(h+kzz)}}$, reperit

$$\int\frac{zzdz}{(h+kzz)^{\frac{3}{2}}\sqrt{(f+gzz)}} = -\frac{1}{k}\int dx\sqrt{\frac{1}{g+(f}}$$

$$\int\frac{dz\sqrt{(f+gzz)}}{(h+kzz)^{\frac{3}{2}}} = -\frac{1}{k}\int dx\sqrt{\frac{g+(fk}{1-}}$$

6. Ponatur $x = \frac{z}{\sqrt{(f+gzz)}}$; erit $dx = \frac{fdz}{(f+gzz)}$

$\sqrt{(f+gzz)} = \frac{\sqrt{f}}{\sqrt{(1-gxx)}}$ et $\sqrt{(h+kzz)} = \sqrt{\frac{h+(fk-gh)}{1-gxx}}$

$$dx\sqrt{\frac{1-gxx}{h+(fk-gh)xx}} = \frac{fdz}{(f+gzz)^{\frac{3}{2}}\sqrt{(h+kzz)}},$$

$$dx\sqrt{\frac{h+(fk-gh)xx}{1-gxx}} = \frac{fdz\sqrt{(h+kzz)}}{(f+gzz)^{\frac{3}{2}}}.$$

$$\int\frac{dz}{(f+gzz)^{\frac{3}{2}}\sqrt{(h+kzz)}} = \frac{1}{f}\int dx\sqrt{\frac{1-gxx}{h+(fk-gh)xx}},$$

$$\int\frac{dz\sqrt{(h+kzz)}}{(f+gzz)^{\frac{3}{2}}} = \frac{1}{f}\int dx\sqrt{\frac{h+(fk-gh)xx}{1-gxx}}.$$

modo ponendo $x=\frac{z}{\sqrt{(h+kzz)}}$ reperitur

$$\int\frac{dz}{(h+kzz)^{\frac{3}{2}}\sqrt{(f+gzz)}} = \frac{1}{h}\int dx\sqrt{\frac{1-kxx}{f+(gh-fk)xx}},$$

$$\int\frac{dz\sqrt{(f+gzz)}}{(h+kzz)^{\frac{3}{2}}} = \frac{1}{h}\int dx\sqrt{\frac{f+(gh-fk)xx}{1-kxx}}.$$

ur $x=\frac{\sqrt{(f+gzz)}}{z}$; erit $dx=\frac{-fdz}{zz\sqrt{(f+gzz)}}$, tum $z=\frac{\sqrt{f}}{\sqrt{(xx-g)}}$,
$\frac{\sqrt{f}x}{\sqrt{(xx-g)}}$ atque $\sqrt{(h+kzz)}=\sqrt{\frac{hxx+fk-gh}{xx-g}}$, unde fit

$$dx\sqrt{\frac{hxx+fk-gh}{xx-g}} = \frac{-fdz\sqrt{(h+kzz)}}{zz\sqrt{(f+gzz)}}.$$

$$dx\sqrt{\frac{xx-g}{hxx+fk-gh}} = \frac{-fdz}{zz\sqrt{(f+gzz)(h+kzz)}},$$

r

$$\int\frac{dz}{zz}\sqrt{\frac{h+kzz}{f+gzz}} = -\frac{1}{f}\int dx\sqrt{\frac{hxx+fk-gh}{xx-g}},$$

$$\int\frac{dz}{zz\sqrt{(f+gzz)(h+kzz)}} = -\frac{1}{f}\int dx\sqrt{\frac{xx-g}{hxx+fk-gh}}.$$

modo ponendo $x=\frac{\sqrt{(h+kzz)}}{z}$ reperietur

$$\int\frac{dz}{zz}\sqrt{\frac{f+gzz}{h+kzz}} = -\frac{1}{h}\int dx\sqrt{\frac{fxx-fk+gh}{xx-k}},$$

$$\int\frac{dz}{zz\sqrt{(f+gzz)(h+kzz)}} = -\frac{1}{h}\int dx\sqrt{\frac{xx-k}{fxx-fk+gh}}.$$

$$\int \frac{dz}{(h+kzz)^{\frac{3}{2}}\sqrt{(f+gzz)}} = \frac{1}{gh-fk}\int dx \sqrt{}$$

11. Simili modo ponendo $x=\sqrt{\frac{h+kzz}{f+gzz}}$ reperitu

$$\int \frac{zzdz}{(f+gzz)^{\frac{3}{2}}\sqrt{(h+kzz)}} = \frac{1}{fk-gh}\int dx \sqrt{}$$

$$\int \frac{dz}{(f+gzz)^{\frac{3}{2}}\sqrt{(h+kzz)}} = \frac{1}{fk-gh}\int dx \sqrt{}$$

COROLLARIUM 1

57. Formulas has in ordinem reducentes, quia ad formam canonicam reducitur, habebimus primo

$$\int \frac{dz}{zz}\sqrt{\frac{f+gzz}{h+kzz}} = -\int dx \sqrt{\frac{fxx+g}{hxx+k}} = -\frac{1}{h}\int dy$$

existente

$$x=\frac{1}{z} \quad \text{et} \quad y=\frac{\sqrt{(h+kzz)}}{z},$$

$$\int \frac{dz}{zz}\sqrt{\frac{h+kzz}{f+gzz}} = -\int dx \sqrt{\frac{hxx+k}{fxx+g}} = -\frac{1}{f}\int dy$$

existente

$$x=\frac{1}{z} \quad \text{et} \quad y=\frac{\sqrt{(f+gzz)}}{z}.$$

COROLLARIUM 2

58. Secunda forma haec esto

$$\int \frac{zzdz}{\sqrt{(f+gzz)}(h+kzz)} = \frac{1}{g}\int dx \sqrt{\frac{xx-f}{gh-fk+kxx}} = \frac{1}{k}$$

existente

$$x=\sqrt{(f+gzz)} \quad \text{et} \quad y=\sqrt{(h+}$$

quae permutatis formulis $\sqrt{(f+gzz)}$ et $\sqrt{(h+kzz)}$ n

COROLLARIUM 3

a forma ita constituatur

$$\ldots gzz) \over \ldots z)^{\frac{3}{2}} = -\frac{1}{k}\int dx\sqrt{\frac{g+(fk-gh)xx}{1-hxx}} = \frac{1}{h}\int dy\sqrt{\frac{f+(gh-fk)yy}{1-kyy}}$$

$$x = \frac{1}{\sqrt{(h+kzz)}} \quad \text{et} \quad y = \frac{z}{\sqrt{(h+kzz)}},$$

$$\ldots kzz) \over \ldots z)^{\frac{3}{2}} = -\frac{1}{g}\int dx\sqrt{\frac{k+(gh-fk)xx}{1-fxx}} = \frac{1}{f}\int dy\sqrt{\frac{h+(fk-gh)yy}{1-gyy}}$$

$$x = \frac{1}{\sqrt{(f+gzz)}} \quad \text{et} \quad y = \frac{z}{\sqrt{(f+gzz)}}.$$

COROLLARIUM 4

a forma haec statuatur

$$\ldots (h+kzz) = -\frac{1}{f}\int dx\sqrt{\frac{xx-g}{hxx+fk-gh}} = -\frac{1}{h}\int dy\sqrt{\frac{yy-k}{fyy-fk+gh}}$$

$$x = \frac{\sqrt{(f+gzz)}}{z} \quad \text{et} \quad y = \frac{\sqrt{(h+kzz)}}{z}.$$

COROLLARIUM 5

a forma erit geminata

$$\ldots dz \over \ldots \sqrt{(h+kzz)} = \frac{1}{f}\int dx\sqrt{\frac{1-gxx}{h+(fk-gh)xx}} = \frac{1}{fk-gh}\int dy\sqrt{\frac{k-gyy}{fyy-h}}$$

$$x = \frac{z}{\sqrt{(f+gzz)}} \quad \text{et} \quad y = \sqrt{\frac{h+kzz}{f+gzz}},$$

$$\ldots dz \over \ldots \sqrt{(f+gzz)} = \frac{1}{h}\int dx\sqrt{\frac{1-kxx}{f+(gh-fk)xx}} = \frac{1}{gh-fk}\int dy\sqrt{\frac{g-kyy}{hyy-f}}$$

$$x = \frac{z}{\sqrt{(h+kzz)}} \quad \text{et} \quad y = \sqrt{\frac{f+gzz}{h+kzz}}.$$

62. Sexta denique forma erit

$$\int \frac{zzdz}{(f+gzz)^{\frac{3}{2}}\sqrt{(h+kzz)}} = -\frac{1}{g}\int dx\sqrt{\frac{1-fxx}{k+(gh-fk)xx}} = \frac{1}{fk-gh}\int d$$

existente

$$x = \frac{1}{\sqrt{(f+gzz)}} \quad \text{et} \quad y = \sqrt{\frac{h+kzz}{f+gzz}}.$$

$$\int \frac{zzdz}{(h+kzz)^{\frac{3}{2}}\sqrt{(f+gzz)}} = -\frac{1}{k}\int dx\sqrt{\frac{1-hxx}{g+(fk-gh)xx}} = \frac{1}{gh-fk}\int dy$$

existente

$$x = \frac{1}{\sqrt{(h+kzz)}} \quad \text{et} \quad y = \sqrt{\frac{f+gzz}{h+kzz}}.$$

PROBLEMA 6

63. *Invenire casus, quibus expressio $\int dz\sqrt{\frac{f+gzz}{h+kzz}}$ aequatur quantit* *$\alpha z\sqrt{\frac{f+gzz}{h+kzz}}$ una cum arcu sectionis conicae.*

SOLUTIO

Ponatur $\int dz\sqrt{\frac{f+gzz}{h+kzz}} = \alpha z\sqrt{\frac{f+gzz}{h+kzz}} + Z$ eritque differentiand

$$dZ = \frac{dz((1-\alpha)fh+(fk+(1-2\alpha)gh)zz+(1-\alpha)gkz^4)}{(h+kzz)^{\frac{3}{2}}\sqrt{(f+gzz)}},$$

ubi numerator neque per $f+gzz$ neque per $h+kzz$ reddi p bilis, quin simul fiat $fk=gh$; reducetur autem Z ad formam § 62 ponendo $\alpha=1$ eritque

$$Z = (fk-gh)\int \frac{zzdz}{(h+kzz)^{\frac{3}{2}}\sqrt{(f+gzz)}}.$$

Hinc habebimus vel

$$\int dz\sqrt{\frac{f+gzz}{h+kzz}} = z\sqrt{\frac{f+gzz}{h+kzz}} + \frac{gh-fk}{k}\int dx\sqrt{\frac{1-hxx}{g+(fk-gh)x}}$$

existente

$$x = \frac{1}{\sqrt{(h+kzz)}}$$

$$\int dz\sqrt{\frac{f+gzz}{h+kzz}} = z\sqrt{\frac{f+gzz}{h+kzz}} - \int dy\sqrt{\frac{hyy-f}{g-kyy}}$$

$$y = \sqrt{\frac{f+gzz}{h+kzz}}.$$

COROLLARIUM 1

...oties ergo vel formula $\int dx\sqrt{\frac{1-hxx}{g+(fk-gh)xx}}$ vel haec $\int dy\sqrt{\frac{hyy-f}{g-kyy}}$ ...m casum iam tractatorum referri potest, toties quoque formula ...$\frac{z}{z}$ partim quantitati algebraicae partim arcui sectionis conicae

COROLLARIUM 2

...um sit $x = \frac{1}{\sqrt{(h+kzz)}}$, erit $1 - hxx = kxxzz$; ergo nisi sit k quan-...va, formula prior non ita, ut fecimus, repraesentari potest. Sci-...it quantitas negativa, ita scribi debet

$$\int dx\sqrt{\frac{hxx-1}{(gh-fk)xx-g}}.$$

COROLLARIUM 3

...n altera formula $\int dy\sqrt{\frac{hyy-f}{g-kyy}}$, ubi $y = \sqrt{\frac{f+gzz}{h+kzz}}$, quia est ...$\frac{(gh-fk)zz}{h+kzz}$, sumitur $gh > fk$. Quare si fuerit $gh < fk$, ea ita scribi ...$\sqrt{\frac{f-hyy}{-g+kyy}}$. Prior scriptio ergo locum habet, si $gh - fk > 0$, ...ero, si $fk - gh > 0$.

EXEMPLUM 1

...educatur forma $\int dx\sqrt{\frac{1-hxx}{g+(fk-gh)xx}}$ ad casum III esseque oportet ...0, $g > 0$ et $fk - gh < 0$, unde $f < 0$, habebiturque

$$\ldots z\sqrt{\frac{-f+gzz}{-h+kzz}} = z\sqrt{\frac{-f+gzz}{-h+kzz}} + \frac{fk-gh}{k}\int dx\sqrt{\frac{1+hxx}{g-(fk-gh)xx}},$$

esse debet $fk > gh$. Iam per § 40 erit

$$\int dx \sqrt{\frac{1+hxx}{g-(fk-gh)xx}} = C - \frac{fk}{(fk-gh)^{\frac{3}{2}}} \Pi^{\frac{fk-gh}{fk}} \left(1 - x\sqrt{\frac{fk-gh}{g}}\right) \left[\frac{fk-}{fk}\right.$$

per § 53

$$\int dx \sqrt{\frac{1+hxx}{g-(fk-gh)xx}} = C + \frac{1}{\sqrt{fk}} \Pi_{\frac{fk}{fk-gh}} \left(1 - \frac{\sqrt{(g-(fk-gh)xx)}}{\sqrt{g}}\right) \left[\frac{f}{fk-}\right.$$

est

$$x = \frac{1}{\sqrt{(-h+kzz)}} \quad \text{et} \quad \sqrt{(g-(fk-gh)xx)} = \frac{\sqrt{k}(gzz-f)}{\sqrt{(-h+kzz)}};$$

que construitur casus XI.

INTEGRATIO CASUS XI

$$\int dz \sqrt{\frac{-f+gzz}{-h+kzz}}$$

$$= C + z\sqrt{\frac{-f+gzz}{-h+kzz}} - \frac{f}{\sqrt{(fk-gh)}} \Pi^{\frac{fk-gh}{fk}} \left(1 - \frac{\sqrt{(fk-gh)}}{\sqrt{g(-h+kzz)}}\right) \left[\frac{fk-g}{fk}\right.$$

$$\int dz \sqrt{\frac{-f+gzz}{-h+kzz}}$$

$$= C + z\sqrt{\frac{-f+gzz}{-h+kzz}} + \frac{fk-gh}{k\sqrt{fk}} \Pi_{\frac{fk}{fk-gh}} \left(1 - \frac{\sqrt{k(-f+gzz)}}{\sqrt{g(-h+kzz)}}\right) \left[\frac{fk}{fk-gh}\right.$$

68. Hoc ergo integrale constat parte algebraica et arcu elliptico, c
et esse $fk > gh$, fieri nequit $= 0$; sin autem sit $h = 0$, ellipsis a
culum atque habebitur

$$\int \frac{dz}{z} \sqrt{\frac{-f+gzz}{k}} = C + \sqrt{\frac{-f+gzz}{k}} - \frac{\sqrt{f}}{\sqrt{k}} \Pi \left(1 - \frac{\sqrt{f}}{z\sqrt{g}}\right) [1]$$

$$\int \frac{dz}{z} \sqrt{\frac{-f+gzz}{k}} = C + \sqrt{\frac{-f+gzz}{k}} + \frac{\sqrt{f}}{\sqrt{k}} \Pi \left(1 - \frac{\sqrt{(-f+gzz)}}{z\sqrt{g}}\right) [1]$$

per integrationem facile invenitur.

Reducatur formula $\int dx \sqrt{\frac{1-hxx}{g+(fk-gh)xx}}$ ad casum VI eritque $h>0$, $gh-fk>0$ et $k>0$; cum autem hoc casu debeat esse $gh-fk>gh$, item f negative capi oportet, ut sit posito $x=\frac{1}{\sqrt{(h+kzz)}}$

$$\int dx \sqrt{\frac{-f+gzz}{h+kzz}} = z\sqrt{\frac{-f+gzz}{h+kzz}} + \frac{gh+fk}{k}\int dx \sqrt{\frac{1-hxx}{g-(fk+gh)xx}}.$$

§ 41 habetur

$$\frac{1-hxx}{g-(fk+gh)xx} = C - \frac{fk}{(fk+gh)^{\frac{3}{2}}}\, \Pi^{\frac{fk+gh}{fk}}\left(1-x\sqrt{\frac{fk+gh}{g}}\right)\left[\frac{fk+gh}{fk}\right],$$

sus IX conficitur.

INTEGRATIO CASUS IX

$$\int dz \sqrt{\frac{-f+gzz}{h+kzz}}$$

$$+ z\sqrt{\frac{-f+gzz}{h+kzz}} - \frac{f}{\sqrt{(fk+gh)}}\, \Pi^{\frac{fk+gh}{fk}}\left(1-\frac{\sqrt{(fk+gh)}}{\sqrt{g(h+kzz)}}\right)\left[\frac{fk+gh}{fk}\right]$$

Casus ergo huius integrale constat parte algebraica et arcu elliptico, semper adhuc alio modo exprimi posset; verum praeferenda est illa cuius axis parametrum superat, ne certis casibus evanescere queat. n hunc casum ex praecedente XI derivare potuissemus ponendo h m, atque si in forma posteriori faciemus $gh>fk$, habebimus aliam onem casus XII.

INTEGRATIO CASUS XII

$$\int dz \sqrt{\frac{-f+gzz}{-h+kzz}}$$

$$+ z\sqrt{\frac{-f+gzz}{-h+kzz}} - \frac{gh-fk}{k\sqrt{fk}}\, \Pi^{\frac{fk}{gh-fk}}\left(\frac{\sqrt{k(-f+gzz)}}{\sqrt{g(-h+kzz)}}-1\right)\left[\frac{-fk}{gh-fk}\right]$$

En aliam integrationem casus XII iam supra § 42 tractati, quae arcum hyperbolicum continet partem algebraicam, cum prior solo

perpendi meretur; quod quo concinnius fiat, ponamus ...

eritque

$$\Pi\frac{m}{n}(t-1)\left|\frac{m}{n}\right| + \Pi\frac{m}{n}\left(\sqrt{\frac{(m+n)tt-m}{(m+n)ntt-n}}-1\right) \ldots$$

$$C+\frac{m}{n}t\sqrt{\frac{(m+n)tt-m}{m(tt-1)}}\,,$$

unde constante debite definita diversi arcus hyperbolici inter ... possunt. Scilicet posito semiaxe $\frac{m}{n}=a$ sumtisque duabus vari...

erit

$$\Pi a(t-1)|a| + \Pi a\left(\sqrt{\frac{(a+1)tt-a}{(a+1)att-1}}-1\right)|a| \ldots$$

$$\Pi a(u-1)|a| \quad \Pi a\left(\sqrt{\frac{(a+1)uu-a}{(a+1)auu-1}}-1\right)|a| \ldots$$

EXEMPLUM 3

72. Ponamus f et k negativa et posterior expressio dat

$$\int dz\sqrt{\frac{f+gzz}{h-kzz}} = z\sqrt{\frac{f+gzz}{h-kzz}} - \int dy\sqrt{\frac{f+hyy}{g-kyy}}$$

existente $gh > fk$. Iam ex casu II § 51 tractato habemus

$$\int dy\sqrt{\frac{f+hyy}{g+kyy}} = C + \frac{f}{\sqrt{(gh-fk)}}\,\Pi\frac{gh-fk}{fk}\left(\sqrt{\frac{g+kyy}{g}}-1\right) \ldots$$

Cum igitur sit

$$y=\sqrt{\frac{f+gzz}{h-kzz}}, \quad \text{erit} \quad \sqrt{(g+kyy)}=\sqrt{\frac{gh-fk}{h-kzz}},$$

unde casus X expeditur.

INTEGRATIO CASUS X

$$\int dz\sqrt{\frac{-f+gzz}{h-kzz}}$$

$$= C + z\sqrt{\frac{-f+gzz}{h-kzz}} - \frac{f}{\sqrt{(gh-fk)}}\, H\, \frac{gh-fk}{fk}\left(\frac{\sqrt{(gh-fk)}}{\sqrt{g(h-kzz)}}-1\right)\left[\frac{-gh+fk}{fk}\right]$$

73. Huius ergo casus X integrale constat parte algebraica et a
rbolico. Sin autem k sumatur negative, oritur integrale casus IX
§ 70 exhibitum, ex quo hic ipse casus derivari potuisset.

EXEMPLUM 4

74. Capiantur g et k negative, ut sit $y=\sqrt{\frac{f-gzz}{h-kzz}}$, eritque

$$\int dz\sqrt{\frac{f-gzz}{h-kzz}} = z\sqrt{\frac{f-gzz}{h-kzz}} - \int dy\sqrt{\frac{hyy-f}{kyy-g}}.$$

dsi forma $\int dy\sqrt{\frac{hyy-f}{kyy-g}}$ hoc modo repraesentetur, ob g et k negative sum
et esse $fk-gh>0$, tum autem non in casu XII continetur, verum
o $\int dy\sqrt{\frac{f-hyy}{g-kyy}}$ repraesentata exigit $gh>fk$, quae conditio casui VI, q
esset referenda, adversatur.

SCHOLION

75. Ope ergo praecedentis problematis casus IX, X et XI sumus exe
ante iam casus III, VI et XII, tum vero etiam II per simplices a
ediverimus. Restant ergo quinque casus nondum realiter resoluti, quo
nullos ita tractare poterimus, ut integrale constet arcu sectionis con
quantitate algebraica formae $z\sqrt{\frac{h+kzz}{f+gzz}}$.

PROBLEMA 7

76. *Invenire casus, quibus expressio* $\int dz\sqrt{\frac{f+gzz}{h+kzz}}$ *aequatur quantitati algebr*
$\sqrt{\frac{h+kzz}{f+gzz}}$ *una cum arcu sectionis conicae.*

SOLUTIO

Ponatur

$$\int dz\sqrt{\frac{f+gzz}{h+kzz}} = \alpha z\sqrt{\frac{h+kzz}{f+gzz}} + Z;$$

$$dZ = \frac{dz(ff - \alpha fh + 2f(g-\alpha k)zz + g(g-\alpha k)z^4)}{(f+gzz)^{\frac{3}{2}}\sqrt{(h+kzz)}}$$

ubi notandum est numeratorem per $f+gzz$ reddi non
quin simul α evanescat. At si ad quandam superiorum for
velimus, poni oportet $\alpha = \frac{g}{k}$, quo facto oritur

$$dZ = \frac{f(fk-gh)}{k}\cdot\frac{dz}{(f+gzz)^{\frac{3}{2}}\sqrt{(h+kzz)}},$$

cuius integratio per § 61 constat. Habebimus ergo vel

$$\int dz\sqrt{\frac{f+gzz}{h+kzz}} = C + \frac{g}{k}z\sqrt{\frac{h+kzz}{f+gzz}} + \frac{fk-gh}{k}\int dx\sqrt{\frac{}{h+}}$$

existente

$$x = \frac{z}{\sqrt{(f+gzz)}}$$

vel

$$\int dz\sqrt{\frac{f+gzz}{h+kzz}} = C + \frac{g}{k}z\sqrt{\frac{h+kzz}{f+gzz}} + \frac{f}{k}\int dy\sqrt{\frac{k-}{fy}}$$

existente

$$y = \sqrt{\frac{h+kzz}{f+gzz}}.$$

COROLLARIUM 1

77. Cum sit $x = \frac{z}{\sqrt{(f+gzz)}}$, erit

$$1 - gxx = \frac{fxx}{zz};$$

quare si fuerit f quantitas positiva, formula recte hoc mod

$$\int dx\sqrt{\frac{1-gxx}{h+(fk-gh)xx}}$$

exprimitur; sin autem sit $f<0$, ita debet repraesentari

$$\int dx\sqrt{\frac{gxx-1}{(gh-fk)xx-h}}.$$

COROLLARIUM 2

Cum sit $y = \sqrt{\frac{h+kzz}{f+gzz}}$, erit

$$fyy - h = \frac{(fk-gh)zz}{f+gzz},$$

formula integralis ita exhibeatur

$$\int dy \sqrt{\frac{k-gyy}{fyy-h}},$$

est sit $fk - gh > 0$; sin autem ita exprimatur

$$\int dy \sqrt{\frac{-k+gyy}{h-fyy}},$$

t $gh - fk > 0$.

EXEMPLUM 1

Referatur forma

$$\int dx \sqrt{\frac{1-gxx}{h+(fk-gh)xx}}$$

III, et quia est $f > 0$, sumi debet $g < 0$, $h > 0$ et $k < 0$, un

$$\sqrt{\frac{f-gzz}{h-kzz}} = C + \frac{g}{k} z \sqrt{\frac{h-kzz}{f-gzz}} + \frac{fk-gh}{k} \int dx \sqrt{\frac{1+gxx}{h-(fk-gk)xx}};$$

ex casu III (§ 40)

$$\frac{1+gxx}{-(fk-gh)xx} = \frac{-fk}{fk-gh} \sqrt{\frac{1}{fk-gh}}\, \Pi^{\frac{fk-gh}{fk}} \left(1 - x\sqrt{\frac{fk-gh}{h}}\right) \left[\frac{fk-gh}{fk}\right.$$

sit $fk > gh$, iterum casus VI occurrit.

INTEGRATIO CASUS VI

$$\int dz \sqrt{\frac{f-gzz}{h-kzz}}$$

$$+ \frac{g}{k} z \sqrt{\frac{h-kzz}{f-gzz}} - \frac{f}{\sqrt{(fk-gh)}}\, \Pi^{\frac{fk-gh}{fk}} \left(1 - \frac{z\sqrt{(fk-gh)}}{\sqrt{h(f-gzz)}}\right) \left[\frac{fk-gh}{fk}\right]$$

Si ellipsin in aliam sui similem invertamus, erit

$$\int dz \sqrt{\frac{f-gzz}{h-kzz}}$$

$$+ \frac{g}{k} z \sqrt{\frac{h-kzz}{f-gzz}} + \frac{fk-gh}{k\sqrt{fk}}\, \Pi^{\frac{fk}{fk-gh}} \left(1 - \frac{\sqrt{f(h-kzz)}}{\sqrt{h(f-gzz)}}\right) \left[\frac{fk}{fk-gh}\right],$$

quod integrale cum superiori § 71 comparatum egregiam [...]
ellipticorum relationem. Sit autem semiaxis

$$\frac{fk}{fk-gh}=a \quad \text{et} \quad z\sqrt{\frac{k}{h}}=t \quad \text{seu} \quad zz=\frac{htt}{k}$$

erit

$$\sqrt{\frac{h-kzz}{f-gzz}}=\sqrt{\frac{h}{f}}\cdot\frac{a(1-tt)}{a-(a-1)tt}$$

ob $gh=\frac{a-1}{a}fk$, unde fit

$$\varPi a(1-t)[a]+\varPi a\left(1-\sqrt{\frac{a(1-tt)}{a-(a-1)tt}}\right)[a]+(a-1)t\sqrt{_a}$$

Sumtis ergo duabus variabilibus t et u habebitur

$$\left.\begin{aligned}&+\varPi a(1-t)[a]+\varPi a\left(1-\sqrt{\frac{a(1-tt)}{a-(a-1)tt}}\right)[a]\\&-\varPi a(1-u)[a]-\varPi a\left(1-\sqrt{\frac{a(1-uu)}{a-(a-1)uu}}\right)[a]\end{aligned}\right\}=\left\{\begin{aligned}&-(a-1\\&+(a-1\end{aligned}\right.$$

unde comparationes arcuum ellipticorum dudum a me d[...]
colliguntur.

Si hic sumatur g negative, oritur casus III et tum fo[...]

$$\int dx\sqrt{\frac{1-gxx}{h-(fk+gh)xx}}$$

ad casum VI referenda fuisset, quare non opus est, ut hunc [...]

EXEMPLUM 2

81. Haec forma nisi invertatur,

$$\int dx\sqrt{\frac{gxx-1}{(gh-fk)xx-h}}$$

ad casum XII reduci nequit, ubi esse debet $f<0$; habebim[...]

$$\int dz\sqrt{\frac{-f+gzz}{h+kzz}}=C+\frac{g}{k}z\sqrt{\frac{h+kzz}{-f+gzz}}-\frac{fk+gh}{k}\int dx\sqrt{_{(}}$$

verum nunc ad casum XI refertur indeque acquireremus
supra inventum.

EXEMPLUM 3

[V]erum formulam

$$\int dx\sqrt{\frac{1-gxx}{h+(fk-gh)xx}}$$

[ad] II reducamus, quod fit sumendo $g<0$ existente $f>0$ et

$$x=\frac{z}{\sqrt{(f-gzz)}},$$

[prodibit]

$$\sqrt{\frac{f-gzz}{h+kzz}}=C-\frac{g}{k}z\sqrt{\frac{h+kzz}{f-gzz}}+\frac{fk+gh}{k}\int dx\sqrt{\frac{1+gxx}{h+(fk+gh)xx}};$$

[Ha]ec reductio non succedit, nisi $k<0$, ita ut sit

$$\sqrt{\frac{f-gzz}{h-kzz}}=C+\frac{g}{k}z\sqrt{\frac{h-kzz}{f-gzz}}-\frac{gh-fk}{k}\int dx\sqrt{\frac{1+gxx}{h+(gh-fk)xx}}$$

$$x=\frac{z}{\sqrt{(f-gzz)}},$$

§ 51

$$\sqrt{\frac{1+gxx}{h+(gh-fk)xx}}=\frac{1}{\sqrt{fk}}\Pi\frac{fk}{gh-fk}\left(\frac{\sqrt{(h+(gh-fk)xx)}}{\sqrt{h}}-1\right)\left[\frac{-fk}{gh-fk}\right]$$

$$\sqrt{(h+(gh-fk)xx)}=\frac{\sqrt{f(h-kzz)}}{\sqrt{(f-gzz)}},$$

[casu]s VII colligitur.

INTEGRATIO CASUS VII

$$\int dz\sqrt{\frac{f-gzz}{h-kzz}}$$

$$+\frac{g}{k}z\sqrt{\frac{h-kzz}{f-gzz}}-\frac{gh-fk}{k\sqrt{fk}}\Pi\frac{fk}{gh-fk}\left(\frac{\sqrt{f(h-kzz)}}{\sqrt{h(f-gzz)}}-1\right)\left[\frac{-fk}{gh-fk}\right]$$

EXISTENTE $gh>fk$

[C]onstat ergo hoc integrale parte algebraica et arcu hyperboli[co] ... [cas]us ad iam expeditos de novo accedit.

SCHOLION

84. Hactenus ergo octo casus per valores reales integravimus, (
II, III, VI, VII, IX, X, XI et XII, et reliqui quatuor ita sunt compa
per similes formas nullo modo integrari queant. Exigunt scilicet
partem algebraicam duos arcus, alterum ellipticum, alterum hyperboli
pars quidem algebraica vel huius formae $z\sqrt{\frac{f+gzz}{h+kzz}}$ vel huius z
assumi potest; unde duo adhuc problemata evolvi conveniet.

PROBLEMA 8

85. *Invenire casus, quibus expressio* $\int dz\sqrt{\frac{f+gzz}{h+kzz}}$ *aequatur quantit*
braicae $\alpha z\sqrt{\frac{f+gzz}{h+kzz}}$ *una cum duobus arcubus sectionum conicarum.*

SOLUTIO

Posito

$$\int dz\sqrt{\frac{f+gzz}{h+kzz}} = \alpha z\sqrt{\frac{f+gzz}{h+kzz}} + Z$$

erit differentiando

$$dZ = \frac{dz\big((1-\alpha)fh + (fk+(1-2\alpha)gh)zz + (1-\alpha)gkz^4\big)}{(h+kzz)^{\frac{3}{2}}\sqrt{(f+gzz)}},$$

quae in duas partes formulis probl. 5. traditis contentas resolvantur.

1. Ponatur

$$Z = p\int\frac{dz}{(h+kzz)^{\frac{3}{2}}\sqrt{(f+gzz)}} + q\int\frac{zz\,dz}{(h+kzz)^{\frac{3}{2}}\sqrt{(f+gzz)}}$$

fierique debet

$$(1-\alpha)fh = p,\quad fk+(1-2\alpha)gh = q,\quad (1-\alpha)gk = 0,$$

unde ob $\alpha = 1$ evanesceret quoque p contra hypothesin.

2. Ponatur

$$Z = p\int\frac{dz}{(h+kzz)^{\frac{3}{2}}\sqrt{(f+gzz)}} + q\int\frac{dz\sqrt{(f+gzz)}}{(h+kzz)^{\frac{3}{2}}}$$

it

$$\alpha)fh = p+qf,\quad fk+(1-2\alpha)gh = qg\quad\text{et}\quad (1-\alpha)gk = 0$$

$$\alpha = 1, \quad q = \frac{fk - gh}{g}, \quad p = \frac{-f(fk - gh)}{g}$$

$$\int dz \sqrt{\frac{f + gzz}{h + kzz}}$$

$$+ z\sqrt{\frac{f + gzz}{h + kzz}} + \frac{f}{g}\int dy \sqrt{\frac{g - kyy}{hyy - f}} - \frac{fk - gh}{gk}\int dx \sqrt{\frac{g + (fk - gh)xx}{1 - hxx}}$$

$$y = \sqrt{\frac{f + gzz}{h + kzz}} \quad \text{et} \quad x = \frac{1}{\sqrt{(h + kzz)}}.$$

natur

$$Z = p\int \frac{dz}{(h + kzz)^{\frac{3}{2}}\sqrt{(f + gzz)}} + q\int \frac{zzdz}{\sqrt{(f + gzz)(h + kzz)}}$$

ecesso est

$$(1 - \alpha)fh = p, \quad fk + (1 - 2\alpha)gh = qh, \quad (1 - \alpha)gk = qk,$$

citur

$$\alpha = \frac{fk}{gh}, \quad q = \frac{gh - fk}{h}, \quad p = \frac{f(gh - fk)}{g}.$$

abebimus

$$\int dz \sqrt{\frac{f + gzz}{h + kzz}}$$

$$+ \frac{fk}{gh} z\sqrt{\frac{f + gzz}{h + kzz}} + \frac{f}{g}\int dy \sqrt{\frac{g - kyy}{hyy - f}} + \frac{gh - fk}{gh}\int dx \sqrt{\frac{xx - f}{gh - fk + kxx}}$$

$$y = \sqrt{\frac{f + gzz}{h + kzz}} \quad \text{et} \quad x = \sqrt{(f + gzz)}.$$

natur

$$Z = p\int \frac{dz}{(h + kzz)^{\frac{3}{2}}\sqrt{(f + gzz)}} + q\int dz \sqrt{\frac{h + kzz}{f + gzz}}$$

ortet

$$\alpha)fh = p + qhh, \quad fk + (1 - 2\alpha)gh = 2qhk, \quad (1 - \alpha)gk = qkk,$$

citur $fk - gh = 0$, quod est absurdum.

5. Ponatur

$$Z = p\int\frac{zzdz}{(h+kzz)^{\frac{3}{2}}\sqrt{(f+gzz)}} + q\int\frac{dz\sqrt{(f+gzz)}}{(h+kzz)^{\frac{1}{2}}};$$

fiet

$$(1-\alpha)fh = qf, \quad fk + (1-2\alpha)gh = p + qg \quad \text{et} \quad (1-\alpha$$

unde nihil ob $q = 0$ concludere licet.

6. Ponatur

$$Z = p\int\frac{zzdz}{(h+kzz)^{\frac{3}{2}}\sqrt{(f+gzz)}} + q\int dz\sqrt{\frac{h+kzz}{f+gzz}}$$

fietque

$$(1-\alpha)fh = qhh, \quad fk + (1-2\alpha)gh = p + 2qhk, \quad (1-\alpha)$$

unde pariter nihil colligi potest.

7. Ponatur

$$Z = p\int\frac{dz\sqrt{(f+gzz)}}{(h+kzz)^{\frac{3}{2}}} + q\int\frac{zzdz}{\sqrt{(f+gzz)(h+kzz)}}$$

eritque

$$(1-\alpha)fh = pf, \quad fk + (1-2\alpha)gh = pg + qh, \quad (1-\alpha)g$$

unde quoque nihil concluditur.

8. Ponatur

$$Z = p\int\frac{dz\sqrt{(f+gzz)}}{(h+kzz)^{\frac{3}{2}}} + q\int dz\sqrt{\frac{h+kzz}{f+gzz}}$$

eritque

$$(1-\alpha)fh = pf + qhh, \quad fk + (1-2\alpha)gh = pg + 2qhk, \quad (1-$$

unde elicitur

$$\alpha = \frac{gh-fk}{gh}, \quad p = \frac{fk-gh}{g}, \quad q = \frac{f}{h}.$$

Quare erit

$$\int dz\sqrt{\frac{f+gzz}{h+kzz}}$$

$$= C + \frac{gh-fk}{gh}z\sqrt{\frac{f+gzz}{h+kzz}} + \frac{f}{h}\int dz\sqrt{\frac{h+kzz}{f+gzz}} + \frac{fk-gh}{gh}\int dy\sqrt{(}$$

existente

$$y = \frac{z}{\sqrt{(h+kzz)}}.$$

Plures combinationes idoneas instituere non licet.

COROLLARIUM 1

Ex hypothesi ultima sponte sequitur integratio casus I, quo e ex casu enim II est

$$dz\sqrt{\frac{h+kzz}{f+gzz}} = \frac{h}{\sqrt{(fk-gh)}}\,\Pi\,\frac{fk-gh}{gh}\left(\frac{\sqrt{(f+gzz)}}{\sqrt{f}}-1\right)\left[\frac{-fk+gh}{gh}\right],$$

casu VI est (§ 41)

$$\int dy\sqrt{\frac{f-(fk-gh)yy}{1-kyy}} = \frac{-gh}{fk}\sqrt{\frac{f}{k}}\,\Pi_{gh}^{fk}(1-y\sqrt{k})\left[\frac{fk}{gh}\right]$$

olligitur

INTEGRATIO CASUS I

$$\int dz\sqrt{\frac{f+gzz}{h+kzz}}$$

$$\frac{fk-gh}{gh}\,z\sqrt{\frac{f+gzz}{h+kzz}}+\frac{f}{\sqrt{(fk-gh)}}\,\Pi\frac{fk-gh}{gh}\left(\frac{\sqrt{(f+gzz)}}{\sqrt{f}}-1\right)\left[\frac{-fk+gh}{gh}\right]$$

$$-\frac{fk-gh}{k\sqrt{fk}}\,\Pi_{gh}^{fk}\left(1-\frac{z\sqrt{k}}{\sqrt{(h+kzz)}}\right)\left[\frac{fk}{gh}\right].$$

COROLLARIUM 2

Ex hypothesi n° 3 casus V deduci posse videtur, unde fit

$$\frac{gzz}{kzz} = \frac{-fk}{gh}\,z\sqrt{\frac{f-gzz}{h+kzz}}-\frac{f}{g}\int dy\sqrt{\frac{g+kyy}{f-hyy}}+\frac{fk+gh}{gh}\int dx\sqrt{\frac{f-xx}{fk+gh-k}}$$

$$y=\sqrt{\frac{f-gzz}{h+kzz}}\quad\text{et}\quad x=\sqrt{(f-gzz)};$$

ultima formula ex casu VI confici nequit neque etiam ex hyp 2.

COROLLARIUM 3

Consideremus formam VIII, ubi g et h sunt negativa, $fk>gh$, atq transferendo habebimus

$$\frac{-gzz}{+kzz} = \frac{fk}{gh}\,z\sqrt{\frac{f-gzz}{-h+kzz}}-\frac{f}{g}\int dy\sqrt{\frac{g+kyy}{f+hyy}}+\frac{gh-fk}{gh}\int dx\sqrt{\frac{f-xx}{fk-gh-k}}$$

existente

$$y = \sqrt{\frac{f - gzz}{-h + kzz}} \quad \text{et} \quad x = \sqrt{(f - gzz)};$$

nunc vero est ex casu II

$$\int dy \sqrt{\frac{g + kyy}{f + hyy}} = \frac{g}{\sqrt{(fk - gh)}} \Pi \frac{fk - gh}{gh} \left(\frac{\sqrt{(f + hyy)}}{\sqrt{f}} - 1 \right) \left[\frac{-fk + g}{gh} \right.$$

existente

$$\sqrt{(f + hyy)} = \frac{z\sqrt{(fk - gh)}}{\sqrt{(-h + kzz)}},$$

deinde ex casu VI

$$\int dx \sqrt{\frac{f - xx}{fk - gh - kxx}} = \frac{-gh}{k\sqrt{fk}} \Pi \frac{fk}{gh} \left(1 - x\sqrt{\frac{k}{fk - gh}} \right) \begin{bmatrix} fk \\ gh \end{bmatrix},$$

unde sequitur

INTEGRATIO CASUS VIII

$$\int dz \sqrt{\frac{f - gzz}{-h + kzz}}$$

$$= C + \frac{fk}{gh} z \sqrt{\frac{f - gzz}{-h + kzz}} - \frac{f}{\sqrt{(fk - gh)}} \Pi \frac{fk - gh}{gh} \left(\frac{z\sqrt{(fk - gh)}}{\sqrt{f(-h + kzz)}} - 1 \right) \left[\frac{-f}{} \right.$$
$$+ \frac{fk - gh}{k\sqrt{fk}} \Pi \frac{fk}{gh} \left(1 - \frac{\sqrt{k(f - gzz)}}{\sqrt{(fk - gh)}} \right) \begin{bmatrix} fk \\ gh \end{bmatrix}.$$

SCHOLION

89. Sic igitur casus duos novos I et VIII sumus adepti, ita ut IV et V supersint, quos ope sequentis problematis superare licebit.

PROBLEMA 9

90. *Invenire casus, quibus expressio $\int dz \sqrt{\frac{f + gzz}{h + kzz}}$ aequatur quantitati al[g] una cum duobus arcubus sectionum conicarum.*

SOLUTIO

$$\int dz \sqrt{\frac{f + gzz}{h + kzz}} = \alpha z \sqrt{\frac{h + kzz}{f + gzz}} + Z$$

rontiando

$$dZ = \frac{dz(ff - \alpha fh + 2f(g - \alpha k)zz + g(g - \alpha k)z^4)}{(f + gzz)^{\frac{3}{2}}\sqrt{(h + kzz)}},$$

olutio in duas partes idoneas sequenti modo instituatur.

onatur

$$Z = p\int\frac{zzdz}{(f + gzz)^{\frac{3}{2}}\sqrt{(h + kzz)}} + q\int\frac{dz\sqrt{(h + kzz)}}{(f + gzz)^{\frac{3}{2}}}$$

$$f(f - \alpha h) = gh, \quad 2f(g - \alpha k) = p + qk, \quad g(g - \alpha k) = 0,$$

ligitur

$$\alpha = \frac{g}{k}, \quad q = \frac{f(fk - gh)}{hk}, \quad p = \frac{-f(fk - gh)}{h}$$

erea

$$\int dz\sqrt{\frac{f + gzz}{h + kzz}}$$

$$+ \frac{gz}{k}\sqrt{\frac{h + kzz}{f + gzz}} - \frac{f}{h}\int dy\sqrt{\frac{fyy - h}{k - gyy}} + \frac{fk - gh}{hk}\int dx\sqrt{\frac{h + (fk - gh)xx}{1 - gxx}}$$

$$y = \sqrt{\frac{h + kzz}{f + gzz}} \quad \text{et} \quad x = \frac{z}{\sqrt{(f + gzz)}}.$$

onatur

$$Z = p\int\frac{zzdz}{(f + gzz)^{\frac{3}{2}}\sqrt{(h + kzz)}} + q\int\frac{zzdz}{\sqrt{(f + gzz)(h + kzz)}}$$

$$f(f - \alpha h) = 0, \quad 2f(g - \alpha k) = p + qf, \quad g(g - \alpha k) = qg$$

$$\alpha = \frac{f}{h}, \quad p = \frac{f(gh - fk)}{h} \quad \text{et} \quad q = \frac{gh - fk}{h};$$

bebitur

$$\frac{gzz}{kzz} = C + \frac{fz}{h}\sqrt{\frac{h + kzz}{f + gzz}} - \frac{f}{h}\int dy\sqrt{\frac{fyy - h}{k - gyy}} + \frac{gh - fk}{gh}\int dx\sqrt{\frac{xx - f}{gh - fk + \ldots}}$$

$$y = \sqrt{\frac{h + kzz}{f + gzz}} \quad \text{et} \quad x = \sqrt{(f + gzz)}.$$

3. Ponatur

$$Z = p\int \frac{zz\,dz}{(f+gzz)^{\frac{3}{2}}\sqrt{(h+kzz)}} + q\int dz\sqrt{\frac{h+k}{f+g}}$$

fietque

$$f(f-\alpha h) = qfh, \quad 2f(g-\alpha k) = p + q(fk+gh), \quad g(g$$

unde nihil concludere licet.

4. Ponatur

$$Z = p\int \frac{dz}{(f+gzz)^{\frac{3}{2}}\sqrt{(h+kzz)}} + q\int \frac{zz\,dz}{\sqrt{(f+gzz)(h+}}$$

fietque

$$f(f-\alpha h) = p, \quad 2f(g-\alpha k) = qf, \quad g(g-\alpha k) =$$

unde nihil concludere licet.

5. Ponatur

$$Z = p\int \frac{dz\sqrt{(h+kzz)}}{(f+gzz)^{\frac{3}{2}}} + q\int \frac{zz\,dz}{\sqrt{(f+gzz)(h+kz}}$$

fietque

$$f(f-\alpha h) = ph, \quad 2f(g-\alpha k) = pk + qf, \quad g(g-\alpha$$

unde nihil colligere licet.

6. Ponatur

$$Z = p\int \frac{dz\sqrt{(h+kzz)}}{(f+gzz)^{\frac{3}{2}}} + q\int dz\sqrt{\frac{h+kzz}{f+gzz}};$$

fieri debet

$$f(f-\alpha h) = ph + qfh, \quad 2f(g-\alpha k) = pk + q(fk+gh), \quad g$$

unde colligitur

$$\alpha = \frac{gh-fk}{hk}, \quad p = \frac{f(fk-gh)}{hk} \quad \text{et} \quad q = \frac{f}{h}$$

ideoque

$$\int dz\sqrt{\frac{f+gzz}{h+kzz}}$$

$$= C + \frac{gh-fk}{hk}\,z\sqrt{\frac{h+kzz}{f+gzz}} + \frac{fk-gh}{hk}\int dy\sqrt{\frac{h+(fk-gh)yy}{1-gyy}} +$$

existente

$$y = \frac{z}{\sqrt{(f+gzz)}}.$$

COROLLARIUM 1

Hinc omnes quatuor casus difficiliores derivari possunt. Primus nempe deducitur ex n^o 6; nam ob $fk > gh$ erit ex casu III

$$\int dy \sqrt{\frac{h + (fk - gh)yy}{1 - gyy}} = -\frac{fk}{g\sqrt{gh}} \Pi \frac{gh}{fk}(1 - y\sqrt{g})\begin{bmatrix} gh \\ -fk \end{bmatrix}$$

$$y = \frac{z}{\sqrt{(f + gzz)}},$$

o ex casu II

$$\int dz \sqrt{\frac{h + kzz}{f + gzz}} = \frac{h}{\sqrt{(fk - gh)}} \Pi \frac{fk - gh}{gh}\left(\frac{\sqrt{(f + gzz)}}{\sqrt{f}} - 1\right)\left[\frac{-fk + gh}{gh}\right] [1]$$

INTEGRATIO CASUS I

$$\frac{+gzz}{+kzz} = C - \frac{fk - gh}{hk} z\sqrt{\frac{h + kzz}{f + gzz}} - \frac{f(fk - gh)}{gh\sqrt{gh}} \Pi \frac{gh}{fk}\left(1 - \frac{z\sqrt{g}}{\sqrt{(f + gzz)}}\right)\begin{bmatrix} gh \\ fk \end{bmatrix}$$

$$+ \frac{f}{\sqrt{(fk - gh)}} \Pi \frac{fk - gh}{gh}\left(\frac{\sqrt{(f + gzz)}}{\sqrt{f}} - 1\right)\left[\frac{-fk + gh}{gh}\right] [1].$$

COROLLARIUM 2

Hic membrum medium per inversionem ellipsis abit in

$$+ \frac{fk - gh}{k\sqrt{fk}} \Pi \frac{fk}{gh}\left(1 - \frac{\sqrt{f}}{\sqrt{(f + gzz)}}\right)\begin{bmatrix} fk \\ gh \end{bmatrix},$$

g negative capiatur, pro casu V manifesto fit pro hyperbola. At negativo erit ultimum membrum ex casu III

$$\int dz \sqrt{\frac{h + kzz}{f - gzz}} = \frac{-(fk + gh)}{gh}\sqrt{\frac{h}{g}} \Pi \frac{gh}{fk + gh}\left(1 - z\sqrt{\frac{g}{f}}\right)\left[\frac{gh}{fk + gh}\right]$$

$$= + \frac{h}{\sqrt{(fk + gh)}} \Pi \frac{fk + gh}{gh}\left(1 - \frac{\sqrt{(f - gzz)}}{\sqrt{f}}\right)\left[\frac{fk + gh}{gh}\right],$$

ucitur

---

[1] ditio princeps: $\Pi \frac{fk - gh}{gh}\left(\frac{\sqrt{(h + kzz)}}{\sqrt{h}} - 1\right)\left[\frac{-fk + gh}{gh}\right]$. Correxit A. K.

INTEGRATIO CASUS V

$$\int dz \sqrt{\frac{f-gzz}{h+kzz}} = C - \frac{fk+gh}{hk} z \sqrt{\frac{h+kzz}{f-gzz}} + \frac{fk+gh}{k\sqrt{fk}}\, \Pi \frac{fk}{gh}\Big(\frac{\cdots}{\sqrt{(\cdots}}$$

$$+ \frac{f}{\sqrt{(fk+gh)}}\, \Pi \frac{fk+gh}{gh}\Big(1-\frac{\sqrt{(f-gzz)}}{\sqrt{f}}\Big)\Big[\frac{fk+\cdots}{gh}$$

COROLLARIUM 3

93. Per n° 2 construitur casus IV, quo h negative ca...

$$\int dz \sqrt{\frac{f+gzz}{-h+kzz}}$$

$$= C - \frac{fz}{h}\sqrt{\frac{-h+kzz}{f+gzz}} + \frac{f}{h}\int dy \sqrt{\frac{h+fyy}{k-gyy}} + \frac{fk+gh}{gh}\int dx \sqrt{\cdots}$$

existente

$$y = \sqrt{\frac{-h+kzz}{f+gzz}} \quad \text{et} \quad x = \sqrt{(f+gzz)}.$$

Nunc vero est

$$\int dy \sqrt{\frac{h+fyy}{k-gyy}} = \frac{-(fk+gh)}{gh}\sqrt{\frac{h}{g}}\, \Pi \frac{gh}{fk+gh}\Big(1-y\sqrt{\cdots}$$

$$= + \frac{h}{\sqrt{(fk+gh)}}\, \Pi \frac{fk+gh}{gh}\Big(1-\frac{\sqrt{(k-gyy)}}{\sqrt{k}}\Big)\Big[\frac{fk\cdots}{g\cdots}$$

existente

$$\sqrt{(k-gyy)} = \frac{\sqrt{(fk+gh)}}{\sqrt{(f+gzz)}}$$

et

$$\int dx \sqrt{\frac{\cdots}{-fk-gh+kxx}} = \frac{gh}{k\sqrt{fk}}\, \Pi \frac{fk}{gh}\Big(x\sqrt{\frac{k}{fk+gh}}-\cdots$$

unde colligitur

INTEGRATIO CASUS IV

$$\int dz \sqrt{\frac{f+gzz}{-h+kzz}}$$

$$= C - \frac{fz}{h}\sqrt{\frac{-h+kzz}{f+gzz}} + \frac{f}{\sqrt{(fk+gh)}}\, \Pi \frac{fk+gh}{gh}\Big(1-\frac{\sqrt{(fk+\cdots}}{\sqrt{k(f+\cdots}}$$

$$+ \frac{fk+gh}{k\sqrt{fk}}\, \Pi \frac{fk}{gh}\Big(\frac{\sqrt{k(f+gzz)}}{\sqrt{(fk+gh)}}-1\Big)\Big[\frac{-fk}{gh}\Big].$$

: insuper g sumamus negative, prodit

INTEGRATIO CASUS VIII

$$\int dz \sqrt{\frac{f-gzz}{-h+kzz}}$$

$$\sqrt{\frac{-h+kzz}{f-gzz}} + \frac{f}{\sqrt{(fk-gh)}} \Pi^{\frac{fk-gh}{gh}} \left(\frac{\sqrt{(fk-gh)}}{\sqrt{k(f-gzz)}} - 1\right) \left[\frac{-fk+gh}{gh}\right]$$
$$+ \frac{fk-gh}{k\sqrt{fk}} \Pi^{\frac{fk}{gh}} \left(1 - \frac{\sqrt{k(f-gzz)}}{\sqrt{(fk-gh)}}\right) \left[\frac{fk}{gh}\right]$$

omnes plane 12 casus expedivimus.

CONCLUSIO

gimus ergo duodecim casus formulae $\int dz \sqrt{\frac{f+gzz}{h+kzz}}$ supra enume-
lasses distingui, quarum quaelibet quatuor casus complectatur.
classis eos continebit casus, quorum integratio simplici arcui
e absolvitur, secunda vero eos, qui insuper partem algebraicam
tertia classis praeter partem algebraicam duos arcus, alterum
erum hyperbolicum, postulat. Cum igitur in enumeratione
c ordinem non respexerimus, iam ita disponendi videntur.

Integralia exprimuntur

arcu elliptico

arcu hyperbolico

parte algebraica et arcu elliptico

parte algebraica et arcu hyperbolico

parte algebraica et duobus arcubus, altero elliptico, altero hyperbolico.

INTEGRATIO AEQUATIONIS

$$\frac{dx}{\sqrt{(A + Bx + Cx^2 + Dx^3 + Ex^4)}} = \frac{dy}{\sqrt{(A + By + Cy^2 +}}$$

Commentatio 345 indicis ENESTROEMIANI
Novi Commentarii academiae scientiarum Petropolitanae 12 (1766/7)
Summarium ibidem p. 5—6

SUMMARIUM

Calculus integralis, ad tantam hodie summorum Geometrarum evectus, insignibus incrementis et subsidiis numquam non ditatus fuit, differentiales solutu difficiliores, quarum integralia casu quasi vel per invenire ipsis licuerat, data opera meditationi subiecerunt methodos eadem, de quibus aliunde iam constitit, integralia perveniendi. Aequ tegrale idque algebraicum et completum via admodum obliqua, cum centra virium fixa attracti motum inquireret, Ill. EULERO invenire li occasione istam integrationem data opera est aggressus eamque su censuit digniorem, quo plura et praeclariora Analyseos artificia difficu plicari videtur, evolutio, cum neutram partem scorsim ne ad arcus logarithmos revocare liceat, polliceri merito videbatur. En igitur dire que substitutionibus et subsidiis analyticis notatu maxime dignis func aequationis integrale eruitur cum priori perfecte congruens; quae cu bus potioribus dubium non sit, quin excoli possit uberius et ad bre nam reduci, ad promovendos Analyseos fines plurimum momenti contine

1. Methodo admodum singulari atque obliqua pervene grationem huius aequationis, cuius integrale idque adeo com

1) L. EULERI Commentationes 251 et 261 (indicis ENESTROEMIANI); v

utrinsque formulae seorsim integrale non solum non algebraice, s
circuli quidem hyperbolaeve quadraturam exprimi potest. Tum ve
imis notatu dignum occurrebat, quod nulla methodus directa p
integrale algebraicum eruendi. Nulla autem occasio magis i
ur fines Analyseos proferendi, quam si, quod methodo obliqua qua
nges elicuerimus, idem methodo directa investigare annitamur.
r nuper[1]) curvas definiverim, quas corpus ad duo centra virium
ctum percurrit, easque ad similem aequationem perduxerim, inde vi
s aequationis integrationem petere licebit; quod quomodo sit praesta
explicare constitui.

2. Ac primo quidem observo aequationem propositam semper in eiu
am transfundi posse, in qua coefficientes B et D evanescant, quod
de alterutro ex elementis satis est notum. Ut autem ambo sim
um redigi queant, id talis formae est proprium; posito enim $x =$
forma, cui quidem altera est similis, abit in hanc

$$\frac{(mb-na)dz}{\ldots nz+b)^4+B(nz+b)^3(mz+a)+C(nz+b)^2(mz+a)^2+D(nz+b)(mz+a)^3+E(mz\ldots}$$

nius denominatore terminos tam ipsa quantitate z quam eius c
os destruere licebit. Prior conditio praebet hanc aequationem

$$\ldots b^3+Bmb^3+3Bnabb+2Cmabb+2Cnaab+3Dmaab+Dna^3+4Em\ldots$$

erior vero hanc

$$\ldots b+Bn^3a+3Bmnnb+2Cmnna+2Cmmnb+3Dmmna+Dm^3b+4Em^3\ldots$$

tam ratio $a:b$ quam ratio $m:n$ elici potest.

1) L. Euleri Commentatio 301 (indicis Enestroemiani): *De motu corporis ad duo ... fixa attracti*, Novi comment. acad. sc. Petrop. 10 (1764), 1766, p. 207; *Leonhardi ... omnia*, series II, vol. 5; Commentatio 328 (indicis Enestroemiani): *De motu corp... ntra virium fixa attracti*, Novi comment. acad. sc. Petrop. 11 (1765), 1767, ... *ardi Euleri Opera omnia*, series II, vol. 5; Commentatio 337 (indicis Enestroe... me. *Un corps étant attiré en raison réciproque quarrée des distances vers deux poin... s, trouver les cas où la courbe décrite par ce corps sera algébrique*, Mém. de l'acad... rlin 16 (1760), 1767, p. 228; *Leonhardi Euleri Opera omnia*, series II, vol. 5.

$$4A + Bq + 3Bp + 2Cpq + 2Cpp + 3Dppq + Dp^3 +$$
$$4A + Bp + 3Bq + 2Cpq + 2Cqq + 3Dpqq + Dq^3 +$$

quarum differentia per $p - q$ divisa praebet

$$2B + 2C(p+q) + D(pp + 4pq + qq) + 4Epq(p +$$

Tum vero prior per q [multiplicata] demta posteriore per
divisione per $p - q$ facta

$$-4A - B(p+q) + Dpq(p+q) + 4Eppqq =$$

statuamus nunc $p + q = r$ et $pq = s$ et ex aequationibus

$$2B + 2Cr + Drr + 2Ds + 4Ers = 0,$$
$$-4A - Br + Drs + 4Ess = 0$$

elidendo $r = \frac{4(A - Ess)}{Ds - B}$ adipiscimur hanc aequationem cubica

$$\left.\begin{matrix} +D^3 \\ -4CDE \\ +8BEE \end{matrix}\right\} s^3 \left.\begin{matrix} -BDD \\ +4BCE \\ -8ADE \end{matrix}\right\} s^2 \left.\begin{matrix} -BBD \\ +4ACD \\ -8ABE \end{matrix}\right\} s \begin{matrix} +B^3 \\ -4AB \\ +8AA \end{matrix}$$

unde incognita s definitur, quod igitur triplici modo fieri p

4. Cum igitur sine detrimento scopi praefixi coefficien
aequales assumere liceat, quaestio nostra in integrali huius
niendo versatur

$$\frac{dx}{\sqrt{(A + Cxx + Dx^4)}} = \frac{dy}{\sqrt{(A + Cyy + Dy^4)}},$$

quam hoc modo repraesentemus

$$\frac{dx}{dy} = \sqrt{\frac{A + Cxx + Dx^4}{A + Cyy + Dy^4}},$$

unde relationem inter variabiles x et y generatim elici
sequenti modo praestare conabor.

$x = \sqrt{npq}$ et $y = \sqrt{n\frac{p}{q}}$, erit

$$dx = \frac{\sqrt{n}(qdp + pdq)}{2\sqrt{pq}} \quad \text{et} \quad dy = \frac{\sqrt{n}(qdp - pdq)}{2q\sqrt{pq}}$$

$$\frac{dx}{dy} = \frac{q(qdp + pdq)}{qdp - pdq}.$$

nu est

$$\frac{A + Cxx + Dx^4}{A + Cyy + Dy^4} = \frac{qq(A + nCpq + nnDppqq)}{Aqq + nCpq + nnDpp},$$

$$\frac{qdp + pdq}{qdp - pdq} = \sqrt{\frac{A + nCpq + nnDppqq}{Aqq + nCpq + nnDpp}},$$

numerus n ad commodum nostrum assumi potest.

t brevitatis gratia

$$\frac{A + nCpq + nnDppqq}{Aqq + nCpq + nnDpp} = \frac{P + Q}{P - Q},$$

$$\frac{(1 + qq) + 2nCpq + nnDpp(1 + qq)}{A(1 - qq) - nnDpp(1 - qq)} = \frac{(A + nnDpp)(1 + qq) + 2nCpq}{(A - nnDpp)(1 - qq)},$$

ob

$$\frac{qdp + pdq}{qdp - pdq} = \sqrt{\frac{P + Q}{P - Q}}$$

us

$$\frac{qdp}{pdq} = \frac{\sqrt{(P + Q)} + \sqrt{(P - Q)}}{\sqrt{(P + Q)} - \sqrt{(P - Q)}} = \frac{P + \sqrt{(PP - QQ)}}{Q}$$

$$\frac{pdq}{qdp} = \frac{P - \sqrt{(PP - QQ)}}{Q}.$$

nne iam momentum versatur in idonea substitutione; atque equidem um observavi

$$q = u + \sqrt{(uu - 1)}, \quad \text{unde fit} \quad \frac{dq}{q} = \frac{du}{\sqrt{(uu - 1)}}$$

$$1 + qq = 2qu, \quad 1 - qq = -2q\sqrt{(uu - 1)},$$

$$\frac{P}{Q} = \frac{(A + nnDpp)u + nCp}{(nnDpp - A)\sqrt{(uu-1)}},$$

ac nunc quidem pro n unitatem commodissime assumi evid
ergo sit

$$\frac{P}{Q} = \frac{(A + Dpp)u + Cp}{(Dpp - A)\sqrt{(uu-1)}},$$

erit

$$\frac{\sqrt{(PP - QQ)}}{Q} = \frac{\sqrt{(4ADppuu + 2Cp(A + Dpp)u + CCpp + (D}}{(Dpp - A)\sqrt{(uu-1)}}$$

ita ut nostra aequatio integranda sit

$$\frac{pdu}{dp} = \frac{(A + Dpp)u + Cp - \sqrt{(4ADppuu + 2Cpu(A + Dpp) + CCpp}}{Dpp - A}$$

8. Ista formula irrationalis hoc modo repraesentetur

$$\sqrt{\left(\left(2pu\sqrt{AD} + \frac{C(A + Dpp)}{2\sqrt{AD}}\right)^2 + \frac{(4AD - CC)(Dpp - A}{4AD}\right.}$$

ac ponatur

$$2pu\sqrt{AD} + \frac{C(A + Dpp)}{2\sqrt{AD}} = \frac{(Dpp - A)s\sqrt{(4AD - CC}}{2\sqrt{AD}}$$

unde fit ipsa formula surda

$$= \frac{(Dpp - A)\sqrt{(4AD - CC)(1 + ss)}}{2\sqrt{AD}}$$

et

$$u = -\frac{C(A + Dpp)}{4ADp} + \frac{(Dpp - A)s\sqrt{(4AD - CC)}}{4ADp}$$

hincque

$$(A + Dpp)u + Cp = \frac{-C(Dpp - A)^2 + (A + Dpp)(Dpp - A)s\sqrt{(}}{4ADp}$$

ita ut iam nostra aequatio sit

$$\frac{pdu}{dp} = \frac{-C(Dpp - A) + (A + Dpp)s\sqrt{(4AD - CC)}}{4ADp} - \frac{\sqrt{(4AD -}}{2\sqrt{}}$$

$$\frac{dp(Dpp-A)}{4ADpp} + \frac{sdp(A+Dpp)\sqrt{(4AD-CC)}}{4ADpp} + \frac{ds(Dpp-A)\sqrt{(4AD-CC)}}{4ADp}$$

ineamus

$$\frac{C(Dpp-A)}{4ADp} + \frac{s(A+Dpp)\sqrt{(4AD-CC)}}{4ADp} + \frac{ds(Dpp-A)\sqrt{(4AD-CC)}}{4ADdp}$$

ula praecedenti aequata commodissime usu venit, ut plerique termi
tollant indeque exsurgat haec aequatio

$$\frac{ds(Dpp-A)\sqrt{(4AD-CC)}}{4ADdp} = \frac{-\sqrt{(4AD-CC)}(1+ss)}{2\sqrt{AD}},$$

citur

$$\frac{ds}{\sqrt{(1+ss)}} = \frac{-2dp\sqrt{AD}}{Dpp-A} = \frac{2dp\sqrt{AD}}{A-Dpp},$$

grale in logarithmis est

$$l\left(s+\sqrt{(1+ss)}\right) = l\frac{\sqrt{A}+p\sqrt{D}}{\sqrt{A}-p\sqrt{D}} + l\alpha,$$

beamus

$$s+\sqrt{(1+ss)} = \frac{\alpha\sqrt{A}+\alpha p\sqrt{D}}{\sqrt{A}-p\sqrt{D}}$$

$$s = \frac{\alpha\alpha(\sqrt{A}+p\sqrt{D})^2-(\sqrt{A}-p\sqrt{D})^2}{2\alpha(A-Dpp)}.$$

Quodsi hinc regrediamur, reperiemus

$$= \frac{-C(A+Dpp)}{4ADp} + \frac{(\sqrt{A}-p\sqrt{D})^2-\alpha\alpha(\sqrt{A}+p\sqrt{D})^2}{8\alpha ADp}\sqrt{(4AD-CC)},$$

niri oportet $q = u+\sqrt{(uu-1)}$. Sed quia hinc fit $u = \frac{1+qq}{2q}$, res
$=xy$ et $q = \frac{x}{y}$ aequatio nostra integralis completa est

$$= \frac{-C(A+Dxxyy)}{4ADxy} + \frac{(\sqrt{A}-xy\sqrt{D})^3-\alpha\alpha(\sqrt{A}+xy\sqrt{D})^2}{8\alpha ADxy}\sqrt{(4AD-CC)}$$

$$= \frac{\sqrt{(4AD - CC)}}{\alpha}\left(\left(\sqrt{A} - xy\sqrt{D}\right)^2 - \alpha\alpha\left(\sqrt{A} + xy\sqrt{D}\right)\right.$$

quae evolvitur in hanc

$$\frac{4AD(xx+yy)+2C(A+Dxxyy)}{\sqrt{(4AD-CC)}} = \frac{(1-\alpha\alpha)A - 2(1+\alpha\alpha)xy\sqrt{AD}}{\alpha}$$

et ponendo

$$\alpha = \frac{\sqrt{(4AD-CC)}}{mC}$$

prodit

$$4AD(xx+yy)+2C(A+Dxxyy)$$
$$= \frac{((1+mm)CC-4AD)(A+Dxxyy) - 2((mm-1)CC+4}{mC}$$

11. Ne casus, ubi $\sqrt{AD}$ fit quantitas imaginaria, turbet tionem alia via, quae ipsa destructione terminorum § 9 ob investigare. Scilicet proposita aequatione

$$\frac{dx}{dy} = \sqrt{\frac{A+Cxx+Ex^4}{A+Cyy+Ey^4}}$$

fiat $x = \sqrt{pq}$ et $y = \sqrt{\frac{p}{q}}$, ut hinc obtineatur

$$\frac{pdq}{qdp} = \frac{P - \sqrt{(PP - QQ)}}{Q}$$

existente

$$\frac{P}{Q} = \frac{(A+Epp)(1+qq)+2Cpq}{(A-Epp)(1-qq)}.$$

Ponatur nunc $q = u + \sqrt{(uu-1)}$, ut sit

$$1+qq = 2qu, \quad 1-qq = 2qu - 2qq = -2q\sqrt{(u$$

erit

$$\frac{dq}{q} = \frac{du}{\sqrt{(uu-1)}} \quad \text{et} \quad \frac{P}{Q} = \frac{u(A+Epp)+C}{(Epp-A)\sqrt{(uu-}}$$

unde resultat haec aequatio transformata

$$\frac{pdu}{dp} = \frac{u(A+Epp)+Cp-\sqrt{(4AEppuu+2Cpu(A+Epp)+C}}{Epp-A}$$

2. Hac aequatione in ordinem redacta et posito brevitatis gratia
ro irrationali $= \sqrt{M}$ fiet

$$udp(A + Epp) + Cpdp - pdu(Epp - A) = dp\sqrt{M}$$

ecto primum hoc membro irrationali reperitur integrale

$$\frac{C + 2Epu}{Epp - A} = \text{Const.};$$

constantis loco autem sumatur quantitas variabilis s, ut sit

$$2Epu + C = s(Epp - A) \quad \text{et} \quad u = \frac{s(Epp - A) - C}{2Ep},$$

hinc membrum rationale fit

$$\frac{-ds(Epp - A)^2}{2E}$$

nula irrationalis

$$(Epp - A)\sqrt{\frac{Ass + Cs + E}{E}},$$

nunc sit

$$\frac{ds}{2}(Epp - A) = dp\sqrt{E(Ass + Cs + E)}$$

$$\frac{ds}{\sqrt{E(Ass + Cs + E)}} + \frac{2dp}{Epp - A} = 0,$$

integrale est

$$\frac{1}{\sqrt{AE}}l\frac{p\sqrt{E} - \sqrt{A}}{p\sqrt{E} + \sqrt{A}} + \frac{1}{\sqrt{AE}}l\Big(As + \frac{1}{2}C + \sqrt{A(Ass + Cs + E)}\Big) = \text{Const.}$$

3. Haec aequatio ergo redit ad hanc formam

$$As + \tfrac{1}{2}C + \sqrt{A(Ass + Cs + E)} = \alpha\frac{p\sqrt{E} + \sqrt{A}}{p\sqrt{E} - \sqrt{A}} = T,$$

elicitur

$$AE = TT - T(2As + C) + \tfrac{1}{4}CC$$

$$+ C = \frac{TT + \frac{1}{4}CC - AE}{T} = \frac{\alpha\alpha(p\sqrt{E} + \sqrt{A})^2 + (\frac{1}{4}CC - AE)(p\sqrt{E} - \sqrt{A})^2}{\alpha(Epp - A)}.$$

Cum nunc sit $p = xy$ et $q = \frac{x}{y}$, erit

$$u = \frac{xx + yy}{2xy} \quad \text{et} \quad s = \frac{E(xx + yy) + C}{Exxyy - A},$$

ex quo efficitur

$$\frac{2AE(xx + yy) + CExxyy + AC}{Exxyy - A} = T + \frac{CC - 4}{4T}$$

existente

$$T = \alpha \cdot \frac{xy\sqrt{E} + \sqrt{A}}{xy\sqrt{E} - \sqrt{A}} = \alpha \cdot \frac{Exxyy + A + 2xy\sqrt{}}{Exxyy - A}$$

et

$$\frac{1}{T} = \frac{1}{\alpha} \cdot \frac{Exxyy + A - 2xy\sqrt{AE}}{Exxyy - A}$$

ideoque

$$2AE(xx + yy) + CExxyy + AC = \alpha(Exxyy + A) +$$
$$+ \frac{CC - 4AE}{4\alpha}(Exxyy + A) - \frac{2(CC - 4AE)}{4\alpha} xy\sqrt{}$$

14. Ne unquam haec expressio involvat imaginaria, co
ita immutemus, ut sit

$$\alpha + \frac{CC - 4AE}{4\alpha} = F \quad \text{seu} \quad 4\alpha\alpha = 4\alpha F - CC +$$

hincque

$$2\alpha = F + \sqrt{(FF + 4AE - CC)} \quad \text{et} \quad \frac{1}{2\alpha} = \frac{F - \sqrt{(FF}}{CC -}$$

unde fit

$$2\alpha - \frac{CC - 4AE}{2\alpha} = 2\sqrt{(FF + 4AE - CC)}$$

et

$$2AE(xx + yy) = (F - C)(Exxyy + A) + 2xy\sqrt{AE(FF}$$

sit nunc $F - C = 2G$; erit

$$AE(xx + yy) = G(A + Exxyy) + 2xy\sqrt{AE(AE +}$$

st aequatio integralis completa huius differentialis

$$\frac{dx}{\sqrt{(A + Cxx + Ex^4)}} = \frac{dy}{\sqrt{(A + Cyy + Ey^4)}},$$

nstans G ita accipi debet, ut formula irrationalis

$$\sqrt{AE(AE + CG + GG)}$$

t imaginaria.

. Forma haec integralis adhuc commodior reddi potest ponendo ff sicque fiet aequatio integralis

$$A(xx + yy) = ff(A + Exxyy) + 2xy\sqrt{A(A + Cff + Ef^4)},$$

st constans arbitraria. Hinc autem elicitur

$$y = \frac{x\sqrt{A(A + Cff + Ef^4)} \pm f\sqrt{A(A + Cxx + Ex^4)}}{A - Effxx}$$

o modo

$$x = \frac{y\sqrt{A(A + Cff + Ef^4)} \pm f\sqrt{A(A + Cyy + Ey^4)}}{A - Effyy}.$$

ormulae cum iis, quas olim[1]) dederam, perfecte consentiunt.

. Integrale hic quidem aequationis differentialis propositae methodo sum consecutus, verumtamen diffiteri non possum hoc per multas es esse praestitum, ita ut vix sit expectandum cuiquam has operationes atem venire potuisse. Ex quo haec ipsa methodus, qua hic sum usus, um in recessu habere videtur neque ullum est dubium, quin eam dili- scrutando aditus ad multa alia praeclara aperiatur ac fortasse alia nethodus idem praestandi detegatur, unde non contemnenda subsidia ad in perficiendam hauriri queant.

. Operationes hic adhibitae aliquantum variari possunt, quod probe disse usu non carebit. Propositam scilicet aequationem differentialem oro

$$\frac{ydx}{xdy} = \sqrt{\frac{Ayy + Cxxyy + Ex^4yy}{Axx + Cxxyy + Exxy^4}} = \sqrt{\frac{P + Q}{P - Q}},$$

$$\frac{P}{Q} = \sqrt{\frac{(A + Exxyy)(xx + yy) + 2Cxxyy}{(A - Exxyy)(yy - xx)}},$$

---

1) L. Euleri Commentatio 261 (indicis Enestroemiani); vide p. 153. A. K.

$$\frac{ydx+xdy}{ydx-xdy} = \frac{\sqrt{(P+Q)}+\sqrt{(P-Q)}}{\sqrt{(P+Q)}-\sqrt{(P-Q)}} =$$

tum etiam

$$\frac{ydx-xdy}{ydx+xdy} = \frac{P-\sqrt{(PP-QQ)}}{Q}.$$

Faciamus nunc hanc substitutionem

$$x = p\left(\sqrt{\frac{q+1}{2}} - \sqrt{\frac{q-1}{2}}\right) \quad \text{et} \quad y = p\left(\sqrt{\frac{q+}{2}}\right.$$

erit

$$xy = pp, \quad xx + yy = 2ppq, \quad yy - xx = 2p$$

deinde

$$\frac{dx}{x} = \frac{dp}{p} - \frac{dq}{2\sqrt{(qq-1)}} \quad \text{et} \quad \frac{dy}{y} = \frac{dp}{p} + \frac{\cdot}{2}$$

unde fit

$$\frac{ydx}{xdy} = \frac{\frac{dp}{p} - \frac{dq}{2\sqrt{(qq-1)}}}{\frac{dp}{p} + \frac{dq}{2\sqrt{(qq-1)}}} \quad \text{et} \quad \frac{ydx-xdy}{ydx+xdy} = \frac{\cdot}{2}$$

atque

$$\frac{P}{Q} = \frac{2(A+Ep^4)ppq + 2Cp^4}{2(A-Ep^4)pp\sqrt{(qq-1)}} = \frac{(A+Ep^4)q}{(A-Ep^4)\sqrt{}}$$

unde fit

$$\frac{\sqrt{(PP-QQ)}}{Q} = \frac{\sqrt{(4AEp^4qq + 2Cppq(A+Ep^4) + C}}{(A-Ep^4)\sqrt{(qq-1}}$$

18. Sit $pp = r$ eritque ob $\frac{dp}{p} = \frac{dr}{2r}$

$$0 = \frac{rdq}{dr} + \frac{(A+Err)q + Cr - \sqrt{(4AErrqq + 2Crq(A+E}}{A-Err}$$

sive

$$rdq(A-Err) + qdr(A+Err) + C$$

$$= dr\sqrt{(4AErrqq + 2Crq(A+Err) + CCrr}$$

Quantitas vinculo radicali implicata ita exhibeatur

$$\frac{1}{4AE}(16AAEErrqq + 8ACErq(A+Err) + 4ACC$$

$$= \frac{1}{4AE}\left((4AErq + C(A+Err))^2 + (4AE - C\right.$$

$$rq + C(A + Err) = s(A - Err)\sqrt{(4AE - CC)}$$

rda

$$= \frac{(A - Err)\sqrt{(4AE - CC)}(1 + ss)}{2\sqrt{AE}}$$

$$s\sqrt{(4AE - CC)} = \frac{4AErq + C(A + Err)}{A - Err}$$

$$= \frac{4AAE(rdq + qdr) - 4AEEr^3dq + 4AEErrqdr + 4ACErdr}{(A - Err)^2}$$

$$) + qdr(A + Err) + Crdr = \frac{ds(A - Err)^2\sqrt{(4AE - CC)}}{4AE};$$

m prius membrum nostrae aequationis, cui aequalis est

$$\frac{dr(A - Err)\sqrt{(4AE - CC)}(1 + ss)}{2\sqrt{AE}},$$

$$\frac{-Err)}{AE} = dr\sqrt{(1 + ss)} \quad \text{et} \quad \frac{2dr\sqrt{AE}}{A - Err} = \frac{ds}{\sqrt{(1 + ss)}},$$

$$s + \sqrt{(1 + ss)} = \alpha \cdot \frac{\sqrt{A} + r\sqrt{E}}{\sqrt{A} - r\sqrt{E}},$$

$$1 = \alpha\alpha\left(\frac{\sqrt{A} + r\sqrt{E}}{\sqrt{A} - r\sqrt{E}}\right)^2 - 2\alpha s \cdot \frac{\sqrt{A} + r\sqrt{E}}{\sqrt{A} - r\sqrt{E}}.$$

$$s = \frac{4AEqr + C(A + Err)}{(A - Err)\sqrt{(4AE - CC)}}$$

$$r = pp = xy \quad \text{et} \quad q = \frac{xx + yy}{2xy}$$

$$s = \frac{2AE(xx + yy) + C(A + Exxyy)}{(A - Exxyy)\sqrt{(4AE - CC)}}.$$

19. Idem expedire possumus sine substitutione nova ... pervenimus ad hanc aequationem

$$rdq(A - Err) + qdr(A + Err) + Crdr$$

$$= dr \sqrt{\frac{(4AErq + C(A + Err))^2 + (4AE - CC)(A - \ldots}{4AE}}$$

notetur esse membrum prius

$$= \frac{(A - Err)^2}{4AE} d. \frac{4AErq + C(A + Err)}{A - Err},$$

posterius vero ita exprimi posse

$$\frac{dr(A - Err)}{2\sqrt{AE}} \sqrt{\left(4AE - CC + \left(\frac{4AErq + C(A + \ldots}{A - Err}\right.\right.}$$

unde posito brevitatis gratia

$$\frac{4AErq + C(A + Err)}{A - Err} = v$$

erit

$$\frac{(A - Err)^2}{4AE} dv = \frac{dr(A - Err)}{2\sqrt{AE}} \sqrt{(4AE - CC + \ldots}$$

ideoque

$$\frac{dv}{\sqrt{(4AE - CC + vv)}} = \frac{2dr\sqrt{AE}}{A - Err}.$$

20. Alind specimen huius reductionis daturus consid... tionem

$$\frac{dx}{\sqrt{(Bx + Cxx + Dx^3)}} = \frac{dy}{\sqrt{(By + Cyy + Dy^3)}}$$

quam ita repraesento

$$\frac{ydx}{xdy} = \sqrt{\frac{Bxyy + Cxxyy + Dx^3yy}{Bxxy + Cxxyy + Dxxy^3}} = \sqrt{\frac{P + \ldots}{P - \ldots}}$$

ut sit

$$\frac{P}{Q} = \frac{Bxy(y + x) + 2Cxxyy + Dxxyy(x + y)}{Bxy(y - x) + Dxxyy(x - y)}$$

seu

$$\frac{P}{Q} = \frac{(B + Dxy)(x + y) + 2Cxy}{(B - Dxy)(y - x)},$$

eritque

$$\frac{ydx - xdy}{ydx + xdy} = \frac{P + \sqrt{(PP - QQ)}}{Q}.$$

21. Statuatur nunc

$$x = p\left(u + \sqrt{(uu-1)}\right) \quad \text{et} \quad y = p\left(u - \sqrt{(uu-1)}\right);$$

$$\frac{dx}{x} = \frac{dp}{p} + \frac{du}{\sqrt{(uu-1)}} \quad \text{et} \quad \frac{dy}{y} = \frac{dp}{p} - \frac{du}{\sqrt{(uu-1)}}$$

uo

$$\frac{ydx - xdy}{ydx + xdy} = \frac{pdu}{dp\sqrt{(uu-1)}}.$$

do ob

$$xy = pp \quad \text{et} \quad x + y = 2pu, \quad y - x = -2p\sqrt{(uu-1)}$$

$$\frac{P}{Q} = \frac{(B + Dpp)u + Cp}{-(B - Dpp)\sqrt{(uu-1)}}$$

que

$$\frac{u}{p} = \frac{(B + Dpp)u + Cp - \sqrt{(4BDppuu + 2Cpu(B + Dpp) + CCpp + (B - Dpp)^2)}}{Dpp - B},$$

e fit

$$udp(B + Dpp) - pdu(Dpp - B) + Cpdp = dp\sqrt{(\ldots)}.$$

ns membrum est

$$(B - Dpp)^2 d. \frac{pu + \frac{C}{4BD}(B + Dpp)}{B - Dpp}$$

$$\frac{(B - Dpp)^2}{4BD} d. \frac{4BDpu + C(B + Dpp)}{B - Dpp},$$

quantitas signo radicali involuta ita scribi potest

$$\frac{1}{BD}\left(16BBDDppuu + 8BCDpu(B + Dpp) + 4BCCDpp + 4BD(B - Dp\right.$$

$$= \frac{1}{4BD}\left((4BDpu + C(B + Dpp))^2 + (4BD - CC)(B - Dpp)^2\right),$$

nde membrum irrationale erit

$$\frac{B - Dpp}{2\sqrt{BD}}\sqrt{\left(4BD - CC + \left(\frac{4BDpu + C(B + Dpp)}{B - Dpp}\right)^2\right)}.$$

$$\frac{4BDpu + C(B + Dpp)}{B - Dpp} = s$$

erit

$$\frac{(B - Dpp)^2}{4BD}ds = \frac{(B - Dpp)dp}{2\sqrt{BD}}\sqrt{(4BD - CC + s\ldots}$$

unde fit

$$\frac{ds}{\sqrt{(4BD - CC + ss)}} = \frac{2dp\sqrt{BD}}{B - Dpp}$$

et integrando

$$s + \sqrt{(4BD - CC + ss)} = \alpha\cdot\frac{\sqrt{B} + p\sqrt{D}}{\sqrt{B} - p\sqrt{D}}$$

ideoque

$$4BD - CC = \alpha\alpha\left(\frac{\sqrt{B} + p\sqrt{D}}{\sqrt{B} - p\sqrt{D}}\right)^2 - 2\alpha s\cdot\frac{\sqrt{B} + p\ldots}{\sqrt{B} - p\ldots}$$

22. Fundamentum ergo harum reductionum in hoc c… ponatur $x = pq$ et $y = \frac{p}{q}$, tum vero pro q eiusmodi formu… partes $x \pm y$, $xx \pm yy$ etc., quae in formula $\frac{P}{Q}$ insunt, qu… reddantur. Veluti in casu § 17 sumsimus

$$q = \sqrt{\frac{u+1}{2}} + \sqrt{\frac{u-1}{2}}$$

seu $qq = u + \sqrt{(uu - 1)}$, in ultimo vero $q = u + \sqrt{(uu - 1)}$… non erat, ut $x + y$ rationaliter exprimatur, unde sufficieba… $u + \sqrt{(uu - 1)}$ tribui, hic vero necesse erat, ut $x + y$ ratio… valorem.

23. Denique casum simpliciorem praetermittere non … ponitur haec aequatio

$$\frac{dx}{\sqrt{(A + Cxx)}} = \frac{dy}{\sqrt{(A + Cyy)}},$$

quam ita refero

$$\frac{ydx}{xdy} = \sqrt{\frac{Ayy + Cxxyy}{Axx + Cxxyy}} = \sqrt{\frac{P + Q}{P - Q}};$$

posito ergo

$$x = p\left(\sqrt{\frac{q+1}{2}} - \sqrt{\frac{q-1}{2}}\right) \quad \text{et} \quad y = p\left(\sqrt{\frac{q+1}{2}} + \ldots\right.$$

$$\frac{-pdq}{2dp\sqrt{(qq-1)}} = \frac{P-\sqrt{(PP-QQ)}}{Q}$$

$$\frac{Aq+Cpp}{A\sqrt{(qq-1)}} \quad \text{et} \quad \frac{\sqrt{(PP-QQ)}}{Q} = \frac{\sqrt{(2ACppq+CCp^4+AA)}}{A\sqrt{(qq-1)}},$$

$pp = r = xy$ erit

$$0 = \frac{rdq}{dr} + \frac{Aq+Cr-\sqrt{(2ACrq+CCrr+AA)}}{A}$$

$$\frac{A(rdq+qdr)+Crdr}{\sqrt{(2ACrq+CCrr+AA)}} = dr,$$

ale est

$$= \sqrt{(2ACrq+CCrr+AA)} \quad \text{seu} \quad FF+2CFr = 2ACrq+AA;$$

$$r = xy \quad \text{et} \quad q = \frac{xx+yy}{2xy},$$

tio integralis est

$$FF+2CFxy = AA+AC(xx+yy).$$

comparatio inter x et y, quae alias per logarithmos vel arcus stendi solet, hic algebraice est eruta.

EVOLUTIO GENERALIOR FORMULARUM COMPARATIONI CURVARUM INSERVIENTIU

Commentatio 347 indicis ENESTROEMIANI
Novi commentarii acad. sc. Petrop. 12 (1766/7), 1768, p. 42—86
Summarium ibidem p. 9—10

SUMMARIUM

Insignia sunt et miro cum ingenii acumine excogitata, quae Ill. Comes FAGNAN
mparatione arcuum curvae lemniscatae elicuit quaeque non minori sagacitate circa
lipticos atque etiam hyperbolicos inter se comparandos est commentatus. Methodum
eometrarum attentione dignissimam iam pridem in hisce Commentariis Ill. EULERUS
editationibus non illustravit modo, sed longe etiam reddidit generaliorem methodu
mendo planam a substitutionibus admodum molestis, quibus FAGNANUS usus est et qu
tio inventionis prorsus est obscura, liberam atque generalissime omnes istorum ar
mparationes in se complexam, cuius ideo beneficio ipsi in gravissimo hoc negotio
ngius progredi licuit. Ad duo vero potissimum capita arduam sane hanc quaesti
vocare licet, dum scilicet demonstravit Cel. EULERUS primo quidem omnium curv
arum rectificatio hac integrali formula contineatur

$$\int \frac{\mathfrak{A}dz}{\sqrt{(A + Cz^2 + Ez^4)}},$$

tque circulares inter se comparari posse, ita ut sumto in istis curvis
alio quovis puncto arcus geometrice abscindi possit, qui ad illum rati
ationalem teneat; deinde vero in curvis, quarum rectificatio ab ista fo

$$\int \frac{dz(\mathfrak{A} + \mathfrak{B}z^2 + \mathfrak{C}z^4 + \mathfrak{D}z^6 + \text{etc.})}{\sqrt{(A + Cz^2 + Ez^4)}}$$

aeque felici successu expediri, quae iam pridem circa comparationem
m praeclara sunt inventa, ita ut in modo memoratis curvis sumto arcu
quovis puncto arcus abscindi possit, qui ab illo vel a quovis eius mul-
fferat vel geometrice assignabili vel a circuli hyperbolaeve quadratura

III. Auctor profundissime huic investigationi incrementum attulit metho-
quoque formulas extendendo, qui expressionem surdam magis complicatam

$$\sqrt{(A+2Bz+Cz^2+2Dz^3+Ez^4)}$$

latissimus aperitur campus in aliis pluribus curvis similes comparationes
rgumentum cum non ad curvarum modo naturam profundius scrutandam
sum, sed largissimam quoque gravissimarum ad Analysin perficiendam
em sistat, in praesenti dissertatione plene evolvitur; cui si addantur ea,
n *Calculi sui integralis* typis in Academia nostra exscripti Vol. I Sect. II
lis formulis integralibus est commentatus, gravissimam quaestionem ad
ncrementum in plena luce positam esse est, quod laetentur Geometrae.

comparatione arcuum circularium ex elementis sunt cognita et
os FAGNANUS de simili comparatione arcuum curvae lemnis-
itato elicuit, ea, uti iam aliquoties[1]) ostendi, ita generalius
, ut, si cuiuspiam lineae curvae arcus indefinite per hanc
lem exprimatur

$$\int\frac{\mathfrak{A}\,dz}{\sqrt{(A+Czz+Ez^4)}},$$

sumto arcu quocunque ab alio quovis puncto arcum geo-
posse illi arcui aequalem. Atque hinc etiam proposito arcu
o quovis puncto arcus abscindi poterit, qui illius arcus sit
ns seu qui in genere ad eum rationem quamcunque ratio-
nde consequitur omnium curvarum, quarum quidem rectifi-
a continetur, arcus perinde atque arcus circulares inter se

1) mmentationes 251, 261, 264 (indicis ENESTROEMIANI); vide p. 58, 153, 201.
A. K.

inventa et quae simili modo ... Comes FAGNANUS circa ...
hyperbolicos summo acumine praestitit, ea deinceps tam ...
stravi, ut pari successu ad omnes curvas, quarum arcus ...
formulam integralem

$$\int \frac{dz(\mathfrak{A} + \mathfrak{B}zz + \mathfrak{C}z^4 + \mathfrak{D}z^6 + \text{etc.})}{\sqrt{(A + Czz + Dz^4)}}$$

exprimatur, extendi queant. Sumto scilicet in tali curva ...
alio quovis puncto arcus abscindi poterit, qui ab illo arcu ...
geometrice assignabili. Tum vero etiam abscindi poteri...
qui ab arcus propositi duplo, triplo vel quovis multiplo ...
geometrice assignabili. Quin etiam illud punctum, unde ar...
ita capi poterit, ut haec differentia plane in nihilum abea...

3. Quaecunque ergo circa arcus parabolicos iam olim s...
quoque in omnibus curvis, quarum rectificatio ad istam f...
est reductibilis, pari successu expediri poterunt. Cum aut...
ad has mirabiles comparationes per substitutiones adm...
quarum ratio inventionis ne quidem perspiciatur, perven...
planam aperui, quae quasi sponte ad easdem comparationes...
ista methodus etiam multo uberius hoc negotium conficit,...
omnes comparationes in se complectitur; aequivalet enim int...
quae simul constantem arbitrariam involvit, dum illae s...
integrationes particulares referre sunt censendae, quam ob ...
huius methodi beneficio multo longius progredi licuit, ...
minibus, quae iam dedi, luculenter apparet.

4. Quemadmodum autem in his formulis, quas pertra...
surda $\sqrt{(A + Czz + Ez^4)}$ implicatur, quae quidem iam ca...
mos complectitur, ita eadem ad expressionem surdam mag...

$$\sqrt{(A + 2Bz + Czz + 2Dz^3 + Ez^4)}$$

extendi posse observavi; qua multo amplior campus aperi...
tiones in pluribus aliis lineis curvis instituendi. Neque...
gatio tantum in lineis curvis tam eximium praestat usum...

alculo integrali gravissima incrementa largiri videtur; ad quae plenius ut viam sternam, evolutiones ad hanc formulam generaliorem perti- ligentius exponam. Hunc in finem proposita sit sequens aequatio n inter binas variabiles x et y exprimens.

AEQUATIO CANONICA EXPENDENDA

$$= \alpha + 2\beta(x + y) + \gamma(xx + yy) + 2\delta xy + 2\varepsilon xy(x + y) + \zeta xxyy$$

aec aequatio praeter binas variabiles x et y continet sex quantitates s, quae autem, cum tantum earum ratio spectetur, ad quinque re- ita ut quinque determinationes ab arbitrio nostro pendentes recipere dum. Deinde etsi haec aequatio ratione variabilium ad quatuor di- s exsurgit, tamen utraque seorsim nusquam ultra duas ascendit, ita que valor per resolutionem aequationis quadraticae exhiberi queat, praesens institutum necessario postulat. Denique ambae variabiles hanc aequationem aequaliter ingrediuntur, et etiamsi permutentur, nutationem inducunt, ut utraque per alteram formula omnino simili r. Atque ob has rationes membra $x^3 + y^3$, $x^4 + y^4$ et $xy(xx + yy)$ iores dimensiones omitti oportuit.

uodsi iam ex hac aequatione tam valorem ipsius x quam ipsius y us, reperiemus

$$= \frac{-\beta - \delta y - \varepsilon yy \pm \sqrt{((\beta + \delta y + \varepsilon yy)^2 - (\alpha + 2\beta y + \gamma yy)(\gamma + 2\varepsilon y + \zeta yy))}}{\gamma + 2\varepsilon y + \zeta yy},$$

$$= \frac{-\beta - \delta x - \varepsilon xx \pm \sqrt{((\beta + \delta x + \varepsilon xx)^2 - (\alpha + 2\beta x + \gamma xx)(\gamma + 2\varepsilon x + \zeta xx))}}{\gamma + 2\varepsilon x + \zeta xx}.$$

brevitatis gratia

$$\pm \sqrt{\left((\beta + \delta y + \varepsilon yy)^2 - (\alpha + 2\beta y + \gamma yy)(\gamma + 2\varepsilon y + \zeta yy)\right)} = Y,$$

$$\pm \sqrt{\left((\beta + \delta x + \varepsilon xx)^2 - (\alpha + 2\beta x + \gamma xx)(\gamma + 2\varepsilon x + \zeta xx)\right)} = X,$$

mus

$$x = \frac{-\beta - \delta y - \varepsilon yy + Y}{\gamma + 2\varepsilon y + \zeta yy} \quad \text{et} \quad y = \frac{-\beta - \delta x - \varepsilon xx + X}{\gamma + 2\varepsilon x + \zeta xx}$$

$$Y = \beta + \delta y + \varepsilon yy + x(\gamma + 2\varepsilon y + \zeta yy),$$

$$X = \beta + \delta x + \varepsilon xx + y(\gamma + 2\varepsilon x + \zeta xx).$$

ifferentialis per binarium divisa

$$0 = + \beta dx + \gamma x dx + \delta y dx + 2\varepsilon xy dx + \varepsilon yy dx + \zeta xyy dx$$
$$+ \beta dy + \gamma y dy + \delta x dy + 2\varepsilon xy dy + \varepsilon xx dy + \zeta xxy dy;$$

uae cum reducatur ad hanc formam

$$0 = + dx(\beta + \delta y + \varepsilon yy) + x dx(\gamma + 2\varepsilon y + \zeta yy)$$
$$+ dy(\beta + \delta x + \varepsilon xx) + y dy(\gamma + 2\varepsilon x + \zeta xx),$$

uoniam coefficientes ipsorum dx et dy sunt eae ipsae quantitates, qua ro formulis radicalibus X et Y exhibuimus, ista aequatio differentia

$$0 = Y dx + X dy \quad \text{seu} \quad \frac{dx}{X} + \frac{dy}{Y} = 0;$$

qua cum variabiles x et y sint separatae, si quidem pro X et Y os surdos substituamus, per integrationem inde hanc aequationem btinebimus

$$\int\frac{dx}{X} + \int\frac{dy}{Y} = \text{Const.}$$

8. Cum igitur haec aequatio integralis certam quandam relationem riabiles x et y exprimat, ea a relatione in aequatione contenta diver n potest sicque ipsa aequatio canonica continebit istam aequationem alem. Etsi ergo in aequatione differentiali $\frac{dx}{X} + \frac{dy}{Y} = 0$ neutra pa tegrabilis atque adeo neque per circuli quadraturam neque logarithm diri potest, tamen integratio algebraicam relationem inter ambas va et y praebet, propterea quod haec aequatio integrata cum ipsa aeq nonica convenit. Quin etiam dico aequationem canonicam non solum rticularem integralis praebere, cuiusmodi casus saepe aequationibus n licatis satisfaciunt, sed eam adeo integrale completum secundum ionem exhibere.

Ad hoc ostendendum, in quo sine dubio summa vis huius integr i debet, notasse sufficit in aequatione canonica una constante ri quam in aequatione differentiali. Vidimus enim aequationem c

involvere constantes arbitrarias, inde examinemus, quot littas-
ntes aequatio differentialis complectatur. Manifestum autem est
odi habere formam

$$\frac{dx}{\sqrt{(A + 2Bx + Cxx + 2Dx^3 + Ex^4)}} + \frac{dy}{\sqrt{(A + 2By + Cyy + 2Dy^3 + Ey^4)}} = 0,$$

dem etiam quinque constantes A, B, C, D, E inesse videntur;
ns est unamquamque per divisionem tolli posse, ita ut re vera
tum inesse sint censendae. Quare cum aequatio integralis quinque
ma arbitrio nostro relinquitur, quod est manifestum indicium
mpleti.

unque autem isti quinque coefficientes A, B, C, D, E se habeant,
icientes aequationis canonicae iis conformiter ita definiri possunt,
neat indeterminatus. Dividamus enim aequationem differentialem
atem indefinitam p, quae iam sublata est censenda, ut re vera

$$X = \sqrt{(Ap + 2Bpx + Cpxx + 2Dpx^3 + Epx^4)}.$$

mus quoque secundum potestates ipsius x valorem primitivum
i erit

$$= \sqrt{\left(\begin{matrix} \beta\beta \\ -\alpha\gamma \end{matrix} \left.\begin{matrix} +2\beta\delta \\ -2\alpha\varepsilon \\ -2\beta\gamma \end{matrix}\right\} x \left.\begin{matrix} +2\beta\varepsilon \\ +\delta\delta \\ -\alpha\zeta \\ -4\beta\varepsilon \\ -\gamma\gamma \end{matrix}\right\} x^2 \left.\begin{matrix} +2\delta\varepsilon \\ -2\beta\zeta \\ -2\gamma\varepsilon \end{matrix}\right\} x^3 \left.\begin{matrix} +\varepsilon\varepsilon \\ -\gamma\zeta \end{matrix}\right\} x^4 \right)},$$

litterae α, β, γ, δ, ε, ζ ita definiantur, ut haec forma cum priori
eddatur; sic enim patebit unam determinationem adhuc arbitrio
qui.

isfieri igitur oportet sequentibus quinque aequationibus

$$\text{I.}\quad \beta\beta - \alpha\gamma = Ap$$

$$\text{II.}\quad \beta\delta - \alpha\varepsilon - \beta\gamma = Bp$$

$$\text{III.}\quad \delta\delta - \alpha\zeta - 2\beta\varepsilon - \gamma\gamma = Cp$$

$$\text{IV.}\quad \delta\varepsilon - \beta\zeta - \gamma\varepsilon = Dp$$

$$\text{V.}\quad \varepsilon\varepsilon - \gamma\zeta = Ep.$$

Ponamus ad abbreviandum $\delta - \gamma = \lambda$ seu $\delta = \gamma + \lambda$ et incipiamus a

$$\text{II.}\ \ \beta\lambda - \alpha\varepsilon = Bp \quad \text{et} \quad \text{IV.}\ \ \varepsilon\lambda - \beta\zeta = Dp,$$

unde definiemus β et ε, ita ut sit

$$\beta = \frac{D\alpha + B\lambda}{\lambda\lambda - \alpha\zeta}p \quad \text{et} \quad \varepsilon = \frac{B\zeta + D\lambda}{\lambda\lambda - \alpha\zeta}p.$$

At I et V coniunctae dant

$$\beta\beta\zeta - \alpha\varepsilon\varepsilon = Ap\zeta - Ep\alpha = \frac{BB\zeta - DD\alpha}{\lambda\lambda - \alpha\zeta}pp,$$

unde eruitur

$$p = \frac{(\lambda\lambda - \alpha\zeta)(A\zeta - E\alpha)}{BB\zeta - DD\alpha},$$

qui valor in alterutra substitutus praebet

$$\gamma = \frac{(A\zeta - E\alpha)((ADD - BBE)\lambda\lambda + 2BD(A\zeta - E\alpha)\lambda + ABB\zeta\zeta - DDE\alpha}{(BB\zeta - DD\alpha)^2}$$

12. Superest igitur III aequatio, quae ob $\delta = \gamma + \lambda$ transit in

$$2\gamma\lambda + \lambda\lambda - \alpha\zeta - 2\beta\varepsilon = Cp.$$

Cum nunc substituto valore ipsius p sit

$$\beta = \frac{(A\zeta - E\alpha)(D\alpha + B\lambda)}{BB\zeta - DD\alpha} \quad \text{et} \quad \varepsilon = \frac{(A\zeta - E\alpha)(B\zeta + D\lambda)}{BB\zeta - DD\alpha},$$

si isti valores pro γ, β, ε et p substituantur, tota aequatio per dividi poterit, quo facto reperietur

$$\lambda = \frac{C(A\zeta - E\alpha)(BB\zeta - DD\alpha) - 2BD(A\zeta - E\alpha)^2 - (BB\zeta - DD\alpha)^2}{2(A\zeta - E\alpha)(ADD - BBE)}.$$

Quoniam igitur nunc omnibus conditionibus est satisfactum, arbitrio adhuc relinquuntur duo coefficientes α et ζ seu potius eorum ratio quam ergo pro lubitu definire licet. Ex quo manifestum est in aeq integrali seu ipsa canonica inesse constantem arbitrariam ab aequatio rentiali non pendentem.

Quia istorum valorum applicatio fieri nequit casibus, quibus

$$ADD - BBE = 0,$$

olutionem huic incommodo non obnoxiam tradam. Posito autem statuo porro

$$\lambda\lambda - \alpha\zeta = \mu \quad \text{seu} \quad \lambda\lambda = \mu + \alpha\zeta$$

nte ex aequationibus II et IV habebimus

$$\beta = \frac{p}{\mu}(D\alpha + B\lambda), \quad \varepsilon = \frac{p}{\mu}(B\zeta + D\lambda).$$

quia I et V coniunctae dant

$$A\zeta - E\alpha = (BB\zeta - DD\alpha)\frac{p}{\mu}.$$

o rationem inter α et ζ, seu quoniam alterutram pro lubitu acci- utramque hoc modo, ut sit

$$\alpha = \mu A - BBp \quad \text{et} \quad \zeta = \mu E - DDp$$

$$\lambda\lambda = \mu + (\mu A - BBp)(\mu E - DDp).$$

a I et V valoribus hactenus inventis substitutis praebebit

$$\gamma = \frac{pp}{\mu\mu}(2BD\lambda + (ADD + BBE)\mu) - \frac{2BBDDp^3}{\mu\mu} - \frac{p}{\mu}.$$

uodsi iam hi valores in aequatione III substituantur, ea ad formam modum prolixam reducitur; verum negotium commodius absolvetur, pro α et ζ inventi in formula ultima praecedentis resolutionis ur; tum enim prodibit

$$\lambda = \frac{\mu\mu}{2p} + BDp - \frac{1}{2}C\mu,$$

ratum cum superiori ipsius $\lambda\lambda$ valore coaequatum praebet

$$Cp)^2 + 4(BD - AE)pp\mu + 4(ADD - BCD + BBE)p^3 = 4pp;$$

$$p = \frac{4}{M(M-C)^2 + 4M(BD-AE) + 4(ADD - BCD + I}$$

et

$$\mu = \frac{4M}{M(M-C)^2 + 4M(BD-AE) + 4(ADD - BCD + I}$$

atque iam M est constans illa arbitraria integrale reddens c

15. Hoc modo omnes coefficientes α, β, γ, δ etc. eode
affecti prodibunt, qui ergo, si per eundem multiplicentur, sec
habebunt

$$\alpha = 4(AM - BB), \quad \beta = 2B(M-C) + 4AD, \quad \gamma = 4AE$$

$$\zeta = 4(EM - DD), \quad \varepsilon = 2D(M-C) + 4BE, \quad \delta = MM - CC$$

ac si illum denominatorem brevitatis gratia statuamus

$$M(M-C)^2 + 4M(BD-AE) + 4(ADD - BCD + E$$

aequatio nostra canonica

$$0 = \alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy + 2\varepsilon xy(x+y)$$

resoluta dabit

$$\beta + \delta x + \varepsilon xx + y(\gamma + 2\varepsilon x + \zeta xx) = \pm 2\sqrt{\Delta(A + 2Bx + Cx}$$

$$\beta + \delta y + \varepsilon yy + x(\gamma + 2\varepsilon y + \zeta yy) = \mp 2\sqrt{\Delta(A + 2By + Cy}$$

simulque est integrale completum huius aequationis different

$$0 = \frac{dx}{\pm\sqrt{(A + 2Bx + Cxx + 2Dx^3 + Ex^4)}} + \frac{dy}{\pm\sqrt{(A + 2By + Cyy}}$$

quia constantem arbitrariam M involvit, quae in aequation
non ingreditur.

INVESTIGATIO CASUUM QUIBUS FORMULA $\frac{Pdx}{X} + \frac{Qdy}{Y}$ FI

16. Designat hic P functionem ipsius x et Q similem
y, et quia haec formula integrabilis esse debet, sit V e

$$\frac{Pdx}{X} + \frac{Qdy}{Y} = dV \quad \text{et} \quad \int \frac{Pdx}{X} + \int \frac{Qdy}{Y} = V.$$

sit

$$\frac{dx}{X} + \frac{dy}{Y} = 0 \quad \text{ideoque} \quad \frac{dy}{Y} = \frac{-dx}{X},$$

$$dV = \frac{(P-Q)dx}{X} = \frac{(P-Q)dx}{\beta + \delta x + \varepsilon xx + y(\gamma + 2\varepsilon x + \zeta xx)}$$

investigari oportet, quibus haec formula integrationem admittit.

oniam vero nulla est ratio, cur hic differentiale dx potius insit quam variabilem introducamus, quae ad utramque aequaliter referatur, uantitas V utramque aequaliter involvere debet. Statuamus ergo t in aequatione differentiali (§ 7) pro dy scribamus $ds - dx$ sic-;

$$\begin{aligned} 0 = &+ dx(\beta + \delta y + \varepsilon yy) + xdx(\gamma + 2\varepsilon y + \zeta yy) \\ &- dx(\beta + \delta x + \varepsilon xx) - ydx(\gamma + 2\varepsilon x + \zeta xx) \\ &+ ds(\beta + \delta x + \varepsilon xx) + yds(\gamma + 2\varepsilon x + \zeta xx), \end{aligned}$$

r ds ita definietur, ut sit

$$dx = \frac{ds(\beta + \delta x + \varepsilon xx) + yds(\gamma + 2\varepsilon x + \zeta xx)}{\delta(x-y) + \varepsilon(xx-yy) - \gamma(x-y) + \zeta xy(x-y)}$$

$$dx = \frac{ds}{x-y} \cdot \frac{\beta + \delta x + \varepsilon xx + y(\gamma + 2\varepsilon x + \zeta xx)}{\delta - \gamma + \varepsilon(x+y) + \zeta xy},$$

substituto fiet

$$dV = \frac{(P-Q)ds}{(x-y)(\delta - \gamma + \varepsilon(x+y) + \zeta xy)}.$$

n P et Q sint similes functiones ipsarum x et y, manifestum est $x-y$ fore divisibile et fractionem $\frac{P-Q}{x-y}$ utramque variabilem x iter esse complexuram. Quia vero posuimus $x + y = s$, ponamus $= t$, ut sit

$$dV = \frac{P-Q}{x-y} \cdot \frac{ds}{\delta - \gamma + \varepsilon s + \zeta t}.$$

$$0 = \alpha + 2\beta s + \gamma ss + 2(\delta - \gamma)t + 2\varepsilon st +$$

ex qua elicitur

$$t = \frac{-\delta + \gamma - \varepsilon s + V((\delta - \gamma)^2 - \alpha\zeta + 2(\delta - \gamma)\varepsilon s - 2\beta\zeta s +}{\zeta}$$

ita ut sit

$$\delta - \gamma + \varepsilon s + \zeta t = V\big((\delta - \gamma)^2 - \alpha\zeta + 2((\delta - \gamma)\varepsilon - \beta\zeta)s$$

Statuamus hanc formulam irrationalem

$$V\big((\delta - \gamma)^2 - \alpha\zeta + 2((\delta - \gamma)\varepsilon - \beta\zeta)s + (\varepsilon\varepsilon - \gamma$$

ut sit

$$t = \frac{-(\delta - \gamma) - \varepsilon s + S}{\zeta} \quad \text{et} \quad dV = \frac{P - Q}{x - y}.$$

19. Ut hinc iam casus integrabilitatis eruamus, pon

$$P = a + bx + cxx + dx^3 + ex^4,$$
$$Q = a + by + cyy + dy^3 + ey^4$$

eritque

$$\frac{P - Q}{x - y} = b + c(x + y) + d(xx + xy + yy) + e(x^3 + x$$

sive introductis novis variabilibus s et t

$$\frac{P - Q}{x - y} = b + cs + d(ss - t) + es(ss - 2$$

At pro t valore substituto habebimus ob $\lambda = \delta - \gamma$

$$\frac{P - Q}{x - y} = b + cs + dss + es^3 + \frac{\lambda d}{\zeta} + \frac{\varepsilon ds}{\zeta} + \frac{2\varepsilon ess}{\zeta} +$$

unde consequimur

$$dV = \frac{\zeta b + \lambda d + (\zeta c + \varepsilon d + 2\lambda e)s + (\zeta d + 2\varepsilon e)ss + \zeta es^3}{\zeta S} d$$

quam formulam integrabilem esse oportet.

...gno hoc melius praestemus, recordemur ex § 13 et 14 esse

$$\ldots - \alpha\zeta = \lambda\lambda - \alpha\zeta = \mu, \quad (\delta-\gamma)\varepsilon - \beta\zeta = Dp \quad \text{et} \quad \varepsilon\varepsilon - \gamma\zeta = Ep,$$

$$S = \sqrt{(\mu + 2Dps + Epss)}$$

... 14 et 15

$$S = \frac{2\sqrt{(M + 2Ds + Ess)}}{\sqrt{\Delta}}.$$

... porro brevitatis gratia

$$b + \frac{\lambda d}{\zeta} = h, \quad c + \frac{\varepsilon d + 2\lambda e}{\zeta} = g, \quad d + \frac{2\varepsilon e}{\zeta} = f,$$

$$dV = \frac{(h + gs + fss + es^3)ds\sqrt{\Delta}}{2\sqrt{(M + 2Ds + Ess)}} - \frac{(d + 2es)ds}{\zeta};$$

... partis prioris integrale

$$(\mathfrak{F} + \mathfrak{G}s + \mathfrak{H}ss)\sqrt{\Delta(M + 2Ds + Ess)}$$

... ferentialium comparatione instituta

$$h = 2\mathfrak{G}M + 2\mathfrak{F}D, \quad g = 4\mathfrak{H}M + 6\mathfrak{G}D + 2\mathfrak{F}E,$$
$$f = 10\mathfrak{H}D + 4\mathfrak{G}E, \quad e = 6\mathfrak{H}E,$$

... integrabilitate requiritur, ut sit

$$\ldots = eD(3EM - 5DD) + fE(3DD - EM) - 2gDEE + 2hE^3.$$

... Hac autem conditione impleta erit

$$\mathfrak{F} = \frac{h}{2D} - \frac{fM}{4DE} + \frac{5eM}{12EE}, \quad \mathfrak{G} = \frac{f}{4E} - \frac{5eD}{12EE}, \quad \mathfrak{H} = \frac{e}{6E}$$

... e quaesitum reperietur

$$V = (\mathfrak{F} + \mathfrak{G}s + \mathfrak{H}ss)\sqrt{\Delta(M + 2Ds + Ess)} - \frac{(d + es)s}{\zeta}$$

$$V = \frac{1}{2}(\mathfrak{F} + \mathfrak{G}s + \mathfrak{H}ss)\Delta S - \frac{(d + es)s}{\zeta}.$$

gralis V ita per x et y exprimetur, ut sit

$$V = \frac{1}{2}\varDelta(\mathfrak{F} + \mathfrak{G}(x+y) + \mathfrak{H}(x+y)^2)(\lambda + \varepsilon(x+y) + \zeta xy) - {}^{d}$$

Quare ut pro V prodeat quantitas algebraica, coefficientes
pro lubitu assumere licet, sed certam quandam relationem
oportet, quae ultima aequalitate paragraphi praecedentis expr
hic assumsi non esse $E=0$; si enim esset $E=0$, valor
algebraice exhiberi posset, uti ex elementis integrationis est

22. Verum si coefficientes b, c, d, e etc. utcunque assu
pressio

$$\int \frac{P dx}{X} + \int \frac{Q dy}{Y}$$

non quidem semper algebraice exhiberi poterit, attamen ei
quadraturam non involvet quam in formula

$$\int \frac{ds}{\sqrt{(M + 2Ds + Ess)}}$$

contentam, quae propterea semper vel per logarithmos v
culares exhiberi poterit. Cum igitur sit

$$X = \sqrt{p}(A + 2Bx + Cxx + 2Dx^3 + Ex^4) \quad \text{et} \quad \sqrt{p}$$

erit

$$X = \frac{2}{\sqrt{\varDelta}}\sqrt{(A + 2Bx + Cxx + 2Dx^3 + Ex^4)}$$

unde invento valore ipsius V habebitur sequens integratio

$$\int \frac{dx(a + bx + cxx + dx^3 + ex^4)}{\sqrt{(A + 2Bx + Cx^2 + 2Dx^3 + Ex^4)}} + \int \frac{dy(a + by + cyy + dy}{\sqrt{(A + 2By + Cyy + 2D}}$$

At substitutis superioribus valoribus erit

$$\frac{2V}{\sqrt{\varDelta}} = \int \frac{\zeta b + \lambda d + (\zeta c + \varepsilon d + 2\lambda e)s + (\zeta d + 2\varepsilon e)ss + \zeta e s^3}{\zeta\sqrt{(M + 2Ds + Ess)}} ds -$$

existente $s = x + y$. Atque hinc sequentia problemata reso

PROBLEMA 1

...re integrale completum huius aequationis differentialis

$$\frac{dy}{\ldots By + Cyy + 2Dy^3 + Ey^4)} = \frac{dx}{\sqrt{(A + 2Bx + Cxx + 2Dx^3 + Ex^4)}}.$$

SOLUTIO

...paret huic aequationi differentiali satisfacere casum $y = x$, qui ...i integrale particulare largitur. Verum ad integrale completum ...quod praeter constantes A, B, C, D, E novam constantem ... involvat, ponamus secundum § 15 brevitatis gratia

$$\ldots - BB), \quad \beta = 2B(M - C) + 4AD, \quad \gamma = 4AE - (M - C)^2,$$

$$\ldots D), \quad \varepsilon = 2D(M - C) + 4BE, \quad \delta = MM - CC + 4(AE + BD)$$

... integralis completa erit

$$\ldots 2\beta(x + y) + \gamma(xx + yy) + 2\delta xy + 2\varepsilon xy(x + y) + \zeta xxyy,$$

... algebraica. Hinc autem sive y per x sive vicissim x per y ... definietur posito item brevitatis ergo

$$\ldots M - C)^2 + 4M(BD - AE) + 4(ADD + BBE) - 4BCD,$$

$$\ldots = \frac{-\beta - \delta x - \varepsilon xx \pm 2\sqrt{\Delta(A + 2Bx + Cxx + 2Dx^3 + Ex^4)}}{\gamma + 2\varepsilon x + \zeta xx}$$

$$\ldots = \frac{-\beta - \delta y - \varepsilon yy \pm 2\sqrt{\Delta(A + 2By + Cyy + 2Dy^3 + Ey^4)}}{\gamma + 2\varepsilon y + \zeta yy},$$

... signorum ambiguorum in utraque expressione vel signa supe-...ora capi debent, ita ut, si in altera formulae surdae tribuatur ... altera formulae surdae signum — tribui debeat. Quae ratio ...ligitur, ubi in aequatione differentiali formulis surdis signa ...adiuncta.

24. Quanquam igitur aequationis differentialis propositae, i
variabiles x et y a se invicem sunt separatae, neutrum m
grationem absolutam admittit atque adeo neque per logarithmos
circulares in genere exprimi potest, tamen vera relatio inter va
aequatione algebraica exhiberi potest.

COROLLARIUM 2

25. Quemadmodum scilicet, si duo arcus quantitate consta
etsi neuter algebraice exprimitur, tamen eorum sinus inter se
tenent rationem, quae satisfacit aequationi differentiali

$$\frac{dy}{\sqrt{(1-yy)}} = \frac{dx}{\sqrt{(1-xx)}},$$

ita quoque aequationis differentialis propositae multoque latius
grale completum algebraice exhiberi potest.

SCHOLION

26. Vis huius solutionis facilius percipietur, si eam ad
restrictos applicemus, inter quos ii praecipue sunt notatu digni
radicale vel unico vel duobus tantum terminis praefigitur, ac si
terminus reperiatur, ratio per se est manifesta.

I. Sit enim $B=0$, $C=0$, $D=0$ et $E=0$, ut integrand

$$\frac{dy}{\sqrt{A}} = \frac{dx}{\sqrt{A}} \quad \text{sive} \quad dy = dx;$$

erit

$$\alpha = 4AM, \quad \beta = 0, \quad \gamma = -MM, \quad \delta = MM, \quad \varepsilon = 0,$$

ideoque aequatio integralis

$$0 = 4AM - MM(xx+yy) + 2MMxy$$

seu

$$x-y = 2\sqrt{\frac{A}{M}} \quad \text{vel} \quad y = x \pm \text{Const.}$$

$= 0$, $C = 0$, $D = 0$ et $E = 0$, ut integranda sit aequatio

$$\frac{dy}{\sqrt{2By}} = \frac{dx}{\sqrt{2Bx}} \quad \text{seu} \quad \frac{dy}{\sqrt{y}} = \frac{dx}{\sqrt{x}};$$

β, $\beta = 2BM$, $\gamma = -MM$, $\delta = MM$, $\varepsilon = 0$ et $\zeta = 0$

io integralis ob $A = M^3$

$$-4BB + 4BM(x+y) - MM(xx+yy) + 2MMxy$$

$$= \frac{2BM - MMx \pm 2\sqrt{2BM^3x}}{-MM} = x + \frac{2B}{M} + 2\sqrt{\frac{2B}{M}x}$$

$\sqrt{x}$ + Const., uti est perspicuum.

$= 0$, $B = 0$, $D = 0$ et $E = 0$, ut integranda sit haec aequatio

$$\frac{dy}{\sqrt{Cyy}} = \frac{dx}{\sqrt{Cxx}} \quad \text{seu} \quad \frac{dy}{y} = \frac{dx}{x};$$

$= 0$, $\gamma = -(M-C)^2$, $\delta = MM - CC$, $\varepsilon = 0$ et $\zeta = 0$

io integralis

$$-(M-C)^2(xx+yy) + 2(MM-CC)xy \quad \text{seu} \quad y = nx.$$

$= 0$, $B = 0$, $C = 0$ et $E = 0$, ut integranda sit haec aequatio

$$\frac{dy}{\sqrt{2Dy^3}} = \frac{dx}{\sqrt{2Dx^3}} \quad \text{seu} \quad \frac{dy}{y\sqrt{y}} = \frac{dx}{x\sqrt{x}};$$

$= 0$, $\gamma = -MM$, $\delta = MM$, $\varepsilon = 2DM$, $\zeta = -4DD$

io integralis

$$MM(xx+yy) + 2MMxy + 4DMxy(x+y) - 4DDxxyy,$$

M^3 dat

$$y = \frac{-MMx - 2DMxx \pm 2\sqrt{2DM^3x^3}}{-MM + 4DMx - 4DDxx}$$

$$\sqrt{y} = \frac{M \pm \sqrt{2DMx}}{M - 2Dx}\sqrt{x} = \frac{\sqrt{Mx}}{\sqrt{M} \pm \sqrt{2D}$$

vel

$$\frac{1}{\sqrt{y}} = \frac{1}{\sqrt{x}} \pm \sqrt{\frac{2D}{M}},$$

uti rei natura postulat.

V. Sit $A = 0$, $B = 0$, $C = 0$ et $D = 0$, ut integr

$$\frac{dy}{\sqrt{Ey^4}} = \frac{dx}{\sqrt{Ex^4}} \quad \text{seu} \quad \frac{dy}{yy} = \frac{dx}{xx};$$

erit

$$\alpha = 0, \quad \beta = 0, \quad \gamma = -MM, \quad \delta = MM, \quad \varepsilon = 0$$

ideoque aequatio integralis

$$0 = -MM(xx + yy) + 2MMxy + 4E$$

hincque

$$y - x = 2xy\sqrt{\frac{E}{M}} \quad \text{seu} \quad \frac{1}{y} = \frac{1}{x} \pm 2$$

Quando autem signum radicale complectitur duos
qui huc pertinent, sequentibus exemplis evolvemus.

EXEMPLUM 1

27. Si sit $C = 0$, $D = 0$ et $E = 0$, ut integranda

$$\frac{dy}{\sqrt{(A + 2By)}} = \frac{dx}{\sqrt{(A + 2Bx)}},$$

invenire aequationem integralem completam.

Erit ergo

$$\alpha = 4(AM - BB), \quad \beta = 2BM, \quad \gamma = -MM, \quad \delta =$$

unde aequatio integralis

$$0 = 4(AM - BB) + 4BM(x + y) - MM(xx +$$

$$= M^3$$

$$= \frac{-2BM - MMx \pm 2\sqrt{M^3(A+2Bx)}}{-MM} = \frac{2B+Mx}{M} \mp 2\sqrt{\frac{A+2Bx}{M}}.$$

...nendo $A = f$, $2B = g$ et $M = c$ sequitur

THEOREMA 1

Huius aequationis differentialis

$$\frac{dy}{\sqrt{(f+gy)}} = \frac{dx}{\sqrt{(f+gx)}}$$

completum est

$$0 = 4cf - gg + 2cg(x+y) - cc(xx+yy) + 2ccxy,$$

$$y = x + \frac{g}{c} \mp 2\sqrt{\frac{f+gx}{c}} \quad \text{et} \quad x = y + \frac{g}{c} \pm 2\sqrt{\frac{f+gy}{c}}.$$

EXEMPLUM 2

Si sit $B = 0$, $D = 0$ et $E = 0$, ut integranda sit aequatio

$$\frac{dy}{\sqrt{(A+Cyy)}} = \frac{dx}{\sqrt{(A+Cxx)}},$$

aequationem integralem completam.

...t ergo

$$...AM, \quad \beta = 0, \quad \gamma = -(M-C)^2; \quad \delta = MM - CC, \quad \varepsilon = 0 \quad \text{et} \quad \zeta = 0,$$

...equatio integralis quaesita erit

$$0 = 4AM - (M-C)^2(xx+yy) + 2(MM-CC)xy,$$

$$...A = M(M-C)^2 \text{ erit}$$

$$\frac{-(MM-CC)x \pm 2(M-C)\sqrt{M(A+Cxx)}}{-(M-C)^2} = \frac{(M+C)x \mp 2\sqrt{M(A+Cxx)}}{M-C}.$$

...ponendo $A = f$, $C = g$ et $M = c$ sequitur

THEOREMA 2

30. *Huius aequationis differentialis*

$$\frac{dy}{\sqrt{(f+gyy)}} = \frac{dx}{\sqrt{(f+gxx)}}$$

integrale completum est

$$0 = 4cf - (c-g)^2(xx+yy) + 2(cc-gg)xy,$$

unde fit

$$y = \frac{(c+g)x \pm 2\sqrt{c(f+gxx)}}{c-g} \quad \text{et} \quad x = \frac{(c+g)y \mp 2\sqrt{c(f+gyy)}}{c-g}$$

EXEMPLUM 3

31. Si sit $B = 0$, $C = 0$ et $E = 0$, ut integranda sit haec ae

$$\frac{dy}{\sqrt{(A+2Dy^3)}} = \frac{dx}{\sqrt{(A+2Dx^3)}},$$

invenire aequationem integralem completam.

Erit ergo

$\alpha = 4AM$, $\beta = 4AD$, $\gamma = -M^2$, $\delta = M^2$, $\varepsilon = 2DM$ et ζ

unde aequatio integralis quaesita est

$$0 = 4AM + 8AD(x+y) - M^2(xx+yy) + 2M^2xy + 4DMxy(x+y) -$$

et cum sit $\Delta = M^2 + 4ADD$, erit

$$y = \frac{-4AD - MMx - 2DMxx \pm 2\sqrt{(M^2+4ADD)(A+2Dx^3}}{-MM+4DMx-4DDxx}$$

sive

$$y = \frac{4AD + MMx + 2DMxx \pm 2\sqrt{(M^2+4ADD)(A+2Dx^3)}}{(M-2Dx)^2}$$

Quare si ponatur $A = f$, $2D = g$ et $M = c$, sequitur

THEOREMA 3

...ius aequationis differentialis

$$\frac{dy}{\sqrt{(f+gy^3)}} = \frac{dx}{\sqrt{(f+gx^3)}}$$

...pletum est

$$+ 4fg(x+y) - cc(xx+yy) + 2ccxy + 2cgxy(x+y) - ggxxyy,$$

$$y = \frac{2fg + ccx + cgxx \pm 2\sqrt{(c^3+fgg)(f+gx^3)}}{(c-gx)^2}$$

$$x = \frac{2fg + ccy + cgyy \mp 2\sqrt{(c^3+fgg)(f+gy^3)}}{(c-gy)^2}.$$

EXEMPLUM 4

...sit $B=0$, $C=0$ et $D=0$, ut aequatio integranda sit

$$\frac{dy}{\sqrt{(A+Ey^4)}} = \frac{dx}{\sqrt{(A+Ex^4)}},$$

...uationem integralem completam.

...go

$\beta = 0$, $\gamma = 4AE - MM$, $\delta = MM + 4AE$, $\varepsilon = 0$ et $\zeta = 4EM$,

...io integralis quaesita est

$$M + (4AE - MM)(xx+yy) + 2(4AE+MM)xy + 4EMxxyy,$$

$\varDelta = M^3 - 4AEM$, erit

$$y = \frac{-(MM+4AE)x \pm 2\sqrt{M(MM-4AE)(A+Ex^4)}}{4AE - MM + 4EMxx}.$$

...natur $A=f$, $E=g$ et $M=2c$, sequitur

THEOREMA 4

34. *Huius aequationis differentialis*

$$\frac{dy}{\sqrt{(f+gy^4)}} = \frac{dx}{\sqrt{(f+gx^4)}}$$

integrale completum est

$$0 = 2cf - (cc - fg)(xx + yy) + 2(cc + fg)xy -$$

unde fit

$$y = \frac{+(cc+fg)x \pm \sqrt{2c(cc-fg)(f+gx^4}}{cc - fg - 2cgxx}$$

et

$$x = \frac{+(cc+fg)y \mp \sqrt{2c(cc-fg)(f+gx^4)}}{cc - fg - 2cgyy}$$

EXEMPLUM 5

35. Si sit $A = 0$, $C = 0$ et $D = 0$, ut integranda

$$\frac{dy}{\sqrt{(2By+Ey^4)}} = \frac{dx}{\sqrt{(2Bx+Ex^4)}},$$

invenire aequationem integralem completam.

Erit ergo

$$\alpha = -4BB, \quad \beta = 2BM, \quad \gamma = -MM, \quad \delta = MM, \quad \varepsilon =$$

hincque aequatio integralis quaesita

$$0 = -4BB + 4BM(x+y) - MM(xx+yy)$$
$$+ 8BExy(x+y) + 4EMxxyy,$$

et cum sit $\Delta = M^3 + 4BBE$, erit

$$y = \frac{2BM + MMx + 4BExx \pm 2\sqrt{(M^3 + 4BBE)}}{MM - 8BEx - 4EMxx}$$

Quare si ponatur $2B = f$, $E = g$, $M = c$, $x = xx$ et y

THEOREMA 5

Huius aequationis differentialis

$$\frac{dy}{\sqrt{(f+gy^6)}} = \frac{dx}{\sqrt{(f+gx^6)}}$$

ompletum est

$$- 2cf(xx+yy) - cc(x^4+y^4) + 2ccxxyy + 4fgxxyy(xx+yy) + 4cgx^4y^4,$$

$$yy = \frac{cf + ccxx + 2fgx^4 \pm 2x\sqrt{(c^3+ffg)(f+gx^6)}}{cc - 4fgxx - 4cgx^4}$$

$$xx = \frac{cf + ccyy + 2fgy^4 \mp 2y\sqrt{(c^3+ffg)(f+gy^6)}}{cc - 4fgyy - 4cgy^4}.$$

SCHOLION 1

robabile hinc videtur etiam huius aequationis differentialis

$$\frac{dy}{\sqrt{(f+gy^n)}} = \frac{dx}{\sqrt{(f+gx^n)}}$$

huius latissime patentis

$$\frac{dy}{\sqrt{(\ldots by + cy^2 + dy^3 + ey^4 + fy^5 + \text{etc.})}} = \frac{dx}{\sqrt{(a + bx + cx^2 + dx^3 + ex^4 + fx^5 + \text{etc.})}},$$

nque dimensiones variabiles x et y in vinculis radicalibus assurgant n dari integralem completam algebraicam. Hoc enim assertum verum est ostensum, quando potestates ipsarum x et y quartum m superant, sed etiam casu $n = 6$, uti vidimus, priorum formu-gratio completa algebraice succedit. Interim tamen nullus adhuc et pro casu $n = 5$ integrale completum aequationis

$$\frac{dy}{\sqrt{(f+gy^5)}} = \frac{dx}{\sqrt{(f+gx^5)}}$$

multo minus id ad casus, quibus n senarium superat, extendere nsi pro casibus $n = 1$, $n = 2$, $n = 3$, $n = 4$ et $n = 6$ sit in promtu. do successu in reliquis casibus vix dubitare licet, tamen restrictio

fractionum adiicere lubuerit, quibus utraque formula per s
uti evenit, si n sit fractio unitatem pro numeratore habens.
certum est veritatem nonnisi pro signo radicali quadrato
neque enim haec aequatio

$$\frac{dy}{\sqrt[3]{(f+gy^3)}}=\frac{dx}{\sqrt[3]{(f+gx^3)}}$$

neque haec

$$\frac{dy}{\sqrt[4]{(f+gy^4)}}=\frac{dx}{\sqrt[4]{(f+gx^4)}}$$

aliaeque harum similes integralia completa algebraica adm
formulae ad rationalitatem perductae tam logarithmos qu
circuli mixtim involvunt atque ex talium quantitatum hete
paratione aequatio algebraica resultare nequit. Haec eadem
tationem superiorem quoque decidit; ac iam audacter pron
hanc aequationem differentialem

$$\frac{dy}{\sqrt{(a+by+cy^2+dy^3+ey^4+fy^5+gy^6)}}=\frac{dx}{\sqrt{(a+bx+cx^2+dx^3+}}$$

generaliter per aequationem algebraicam integrari non poss
quaeretur integratio algebraica huius aequationis

$$\frac{dy}{A+By+Cyy+Dy^3}=\frac{dx}{A+Bx+Cxx+Dx^3},$$

quod utique esset absurdum; multo minus igitur integratio
magis compositis succedet. Verum nequidem integrabilita
quintam usque extendi potest; nam posito $y=0$ si etiam
et pro y et x scribatur yy et xx, prodit haec aequatio diff

$$\frac{dy}{\sqrt{(b+cy^2+dy^4+ey^6+fy^8)}}=\frac{dx}{\sqrt{(b+cx^2+dx^4+ex^6+}}$$

in qua, si radicis extractio succedat, continebitur haec

$$\frac{dy}{A+By^2+Cy^4}=\frac{dx}{A+Bx^2+Cx^4},$$

quam in genere integrationem algebraicam non admittere c

Hunc igitur pro certo affirmare licet ex hoc genere aequationem em latissime patentem, quae quidem generaliter algebraice integrari e eam ipsam, quam hactenus tractavimus

$$\frac{dy}{\sqrt{(A+2By+Cy^2+2Dy^3+Ey^4)}}=\frac{dx}{\sqrt{(A+2Bx+Cx^2+2Dx^3+Ex^4)}}$$

equationem integralem completam assignavimus. Quam ob causam atio multo magis est notatu digna, quod in hoc genere est generaae integrationem algebraicam admittat. Quoniam igitur eius inteiam exposui, operae pretium erit eius usum in comparatione linearum, quarum elementa per huiusmodi formulas exprimuntur, uberius si quidem in iis omnia continentur, quae in hoc genere praestari Atque haec ipsa consideratio nos quoque ad integrationem huiusmodi m

$$\frac{ndy}{\sqrt{(A+2By+Cy^2+2Dy^3+Ey^4)}}=\frac{mdx}{\sqrt{(A+2Bx+Cx^2+2Dx^3+Ex^4)}}$$

, si quidem m et n fuerint numeri integri.

PROBLEMA 2

linea curva habeatur, cuius arcus sive abscissae sive applicatae sive alii cuicunque rectae variabili z ad curvam relatae respondens sit

$$\frac{\mathfrak{A}dz}{\sqrt{(\ldots+2Bz+Cz^2+2Dz^3+Ez^4)}},$$

hac curva arcus quicunque *1), ab alio quovis puncto P* *ndere PQ, qui aequalis sit illi*

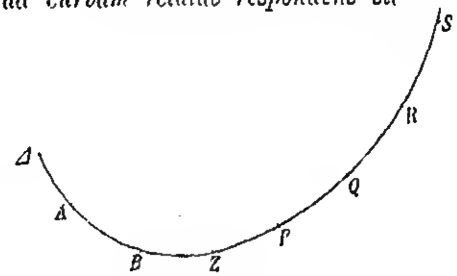


Fig. 1

SOLUTIO

efficientibus datis A, B, C, D, E quaerantur hi alii

$$AM-BA),\quad \beta=2B(M-C)+4AD,\quad \gamma=4AE-(M-C)^2,$$

$$-DD),\quad \varepsilon=2D(M-C)+4BE,\quad \delta=MM-CC+4(AE+BD),$$

aequationem algebraicam

$$0 = \alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy + 2\varepsilon xy(x+$$

congruere cum hac transcendente

$$\int \frac{\mathfrak{A}dy}{\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}} - \int \frac{\mathfrak{A}dx}{\sqrt{(A+2Bx+Cxx+2D}}$$

ubi quantitas constans ita definiri debet, ut illi M sit co
ponamus in curva proposita variabilem z puncto Z res
initium in puncto $\varDelta$ statui atque ad abbreviandum hunc
cemus $\Pi:z$, ut sit

$$\int \frac{\mathfrak{A}dz}{\sqrt{(A+2Bz+Czz+2Dz^3+Ez^4)}} = \Pi:z,$$

erit ex aequatione superiori

$$\Pi:y - \Pi:x = \text{Const.}$$

Respondeant nunc punctis A et B rectae a et b, punctis
p et q, ut sint arcus

$$\varDelta A = \Pi:a, \quad \varDelta B = \Pi:b, \quad \varDelta P = \Pi:p \quad \text{et} \quad \varDelta$$

ideoque

$$\text{arcus } AB = \Pi:b - \Pi:a \quad \text{et} \quad \text{arcus } PQ = \Pi:$$

ac loco x et y scribamus p et q, ut sit

$$0 = \alpha + 2\beta(p+q) + \gamma(pp+qq) + 2\delta pq + 2\varepsilon pq(p -$$

erit $\Pi:q - \Pi:p = \text{Const.}$ Quodsi ergo constantem M
facto $p=a$ prodeat $q=b$, habebimus

$$\Pi:q - \Pi:p = \Pi:b - \Pi:a$$

ideoque arcum $PQ =$ arcui AB, uti requiritur. Constan
ponamus $M - C = L$, ut sit $M = C + L$, constans L ex
debet definiri

$$\begin{aligned} 0 = 4AC - 4BB + 4AL + 2(2BL + 4AD)(a+b) + (4A \\ + 2(LL + 2CL + 4AE + 4BD)ab + 2(2DL + 4B \\ + 4(CE - DD + EL)aabb, \end{aligned}$$

$$\ldots(A+B(a+b)+Cab+Dab(a+b)+Eaabb)+\frac{4AC+8AD(a+b)+8(AE+BD)ab}{(b-a)^2}$$
$$+\frac{4CEaabb-4BB+4AE(aa+bb)+8BEab(a+b)-4DDaabb}{(b-a)^2}\Bigg\}$$

extracta

$$\left\{\begin{array}{c}\dfrac{2(A+B(a+b)+Cab+Dab(a+b)+Eaabb)}{(b-a)^2}\\[2ex] \pm\dfrac{2\sqrt{(A+2Ba+Caa+2Da^3+Ea^4)(A+2Bb+Cbb+2Db^3+Eb^4)}}{(b-a)^2}\end{array}\right\}$$

t

$$M=\frac{2A+2B(a+b)+C(aa+bb)+2Dab(a+b)+2Eaabb}{(b-a)^2}$$

$$\ldots\sqrt{(A+2Ba+Caa+2Da^3+Ea^4)(A+2Bb+Cbb+2Db^3+Eb^4)}.$$

ro invento si iam definiantur valores coefficientium α, β, γ, δ, ε, ζ, ex dato curvae puncto P datur variabilis p, ex ea valor idoneus q, cui curvae punctum Q respondet, determinabitur per hanc om

$$= \alpha+\beta(p+q)+\gamma(pp+qq)+2\delta pq+2\varepsilon pq(p+q)+\zeta ppqq;$$

i brevitatis gratia ponamus

$$= M(M-C)^2+4M(BD-AE)+4(ADD+BBE)-4BCD,$$

$$q=\frac{-\beta-\delta p-\varepsilon pp\pm 2\sqrt{\Delta(A+2Bp+Cpp+2Dp^3+Ep^4)}}{\gamma+2\varepsilon p+\zeta pp}$$

to arcu AB et puncto P assignabitur punctum Q, ut arcus PQ at arcui AB. Reperientur autem ob signum ambiguum bina puncta altorum citra, alterum ultra punctum P erit situm.

COROLLARIUM 1

avento valore q simili modo a puncto Q ulterius abscindi poterit arcui AB aequalis. Posita enim variabili puncto R respondente

sicque a puncto P simul abscindetur arcus PR duplus arcus

COROLLARIUM 2

41. Quoniam r hinc duplicem obtinet valorem, notandu
iterum in p abire, quia ante animadvertimus esse

$$p = \frac{-\beta - \delta q - \varepsilon qq \mp 2\sqrt{\Delta(A + 2Bq + Cqq + 2Dq^3 + E}{\gamma + 2\varepsilon q + \zeta qq}$$

quare, ut arcus PR evadat duplus, idem signum, quod in
fuerit electum, in valore ipsius r capi oportet.

COROLLARIUM 3

42. Pari modo ultra R reperietur punctum S, ut de
aequalis sicque angulus PS triplus evadat arcus AB; inventa
r valor variabilis s puncto S respondentis hac formula exprim

$$s = \frac{-\beta - \delta r - \varepsilon rr \pm 2\sqrt{\Delta(A + 2Br + Crr + 2Dr^3 + E}{\gamma + 2\varepsilon r + \zeta rr}$$

hocque modo quousque libuerit ulterius progredi licet.

COROLLARIUM 4

43. Hac ergo repetita operatione a dato puncto P arcus
qui se habeat ad arcum AB, ut numerus quicunque integer
Quare si ab alio puncto abscindatur arcus, qui sit ad eunde
numerus integer n ad unitatem, duo habebuntur arcus ration
numeri ad numerum tenentes.

COROLLARIUM 5

44. Omnium igitur curvarum, quarum arcus variabili cu
deus huiusmodi formula

$$\int \frac{\mathfrak{A}\,dz}{\sqrt{(A + 2Bz + Czz + 2Dz^3 + Ez^4)}}$$

quo arcus circuli inter se comparare licet. Atque ob rationes s haec similitudo cum circulo vix ad alias curvas, nisi quarum hanc formulam reduci potest, extendi videtur.

EXEMPLUM

osita sit linea curva, cuius arcus ad quampiam rectam variabilem v rmula integrali $\int \frac{dv}{\sqrt{(1-v^6)}}$ exprimatur, cuiusmodi curvae algebraicae peri possunt, in qua a puncto P arcus abscindi oporteat PQ, atum arcum AB rationem tenentes vel aequalitatis vel duplam

c expressio in nostra forma generali non continetur, eo reducatur z seu $v = \sqrt{z}$; sic enim arcus huic novae variabili z respondens $\frac{}{-z^4)}$. Fiat ergo $\mathfrak{A} = \frac{1}{2}$ et $A = 0$, $B = \frac{1}{2}$, $C = 0$, $D = 0$ et lo obtinetur

$$\beta = M, \quad \gamma = -MM, \quad \delta = MM, \quad \varepsilon = -2, \quad \zeta = -4M$$

ituta aequatione

$$M(p+q) - MM(pp+qq) + 2MMpq - 4pq(p+q) - 4Mppqq,$$

$$q = \frac{M + MMp - 2pp \pm 2\sqrt{(M^3-1)(p-p^4)}}{MM + 4p + 4Mpp},$$

$$\int \frac{dq}{2\sqrt{(q-q^4)}} - \int \frac{dp}{2\sqrt{(p-p^4)}} = \text{Const.}$$

$$\Pi : q - \Pi : p = \Pi : b - \Pi : a,$$

b, p, q sint valores variabilis z, qui arcubus $\varDelta A$, $\varDelta B$, $\varDelta P$ et nt. At iam constans M ex datis a et b ita definiri debet, ut sit

$$-1 + 2M(a+b) - MM(b-a)^2 - 4ab(a+b) - 4Maabb,$$

$$M = \frac{a + b - 2aabb \pm 2\sqrt{(a - a^4)(b - b^4)}}{(b-a)^2}$$

et

$$\sqrt{(M-1)} = \frac{\sqrt{a(1-a)(1+b+bb)} \pm \sqrt{b(1-b)(1+a+aa)}}{(b-a)},$$

$$\sqrt{(M^3-1)} = \frac{(a+3b-4ab^3)\sqrt{(a-a^4)} + (b+3a-4a^3b)\sqrt{(b-b^4)}}{(b-a)^3}.$$

Invento hoc modo valore constantis M ex data quantitate p invenit hinc porro valor variabilis r puncto R respondens, scilicet

$$r = \frac{M + MMq - 2qq \pm 2\sqrt{(M^3-1)(q-q^4)}}{MM + 4q + 4Mqq},$$

sicque a puncto P arcus quicunque multiplus arcus dati AB abscin

SCHOLION

46. Circa huiusmodi curvas singularis affectio notari meretur brevitatis gratia ponamus

$$\sqrt{(a-a^4)} = \mathfrak{a} \quad \text{et} \quad \sqrt{(b-b^4)} = \mathfrak{b},$$

ut sit

$$M = \frac{a + b - 2aabb + 2\mathfrak{a}\mathfrak{b}}{(b-a)^2} \quad \text{et} \quad \sqrt{(M^3-1)} = \frac{(a+3b-4ab^3)\mathfrak{a} + (b+3a}{(b-a)^3}$$

utraque quantitas radicalis $\mathfrak{a}$ et $\mathfrak{b}$ tam affirmative quam negative ca unde pro M geminus valor habetur; ex quo pro

$$q = \frac{M + MMp - 2pp \pm 2\sqrt{(M^3-1)(p-p^4)}}{MM + 4p + 4Mpp}$$

ob novam signi ambiguitatem quaterni valores resultant. Binc natura rei ostendit, quia punctum Q tam ante quam post puncta potest, sed quia quatuor reperiuntur, id indicio est curvam duplici praeditam et in utroque arcus aequales exhiberi. Consideremus ca punctum P in ipso puncto A capitur, ita ut sit $p = a$ et

$$q = \frac{M + MMa - 2aa \pm 2\mathfrak{a}\sqrt{(M^3-1)}}{MM + 4a + 4Maa},$$

o forma substituto pro M valoro statim duos valores praebet aequale
b; at duo roliqui divorsi continentur in

$$\frac{4a^3+9aab-6abb+b^3-4a^6-12a^4bb+8a^6b^3\pm 4a(3a-b-2a^3b)ab}{aa+6ab+bb+8a^5-24a^4b+16a^3bb-16aab^3+16a^6b^3-8a^6bb\pm 4(a+b-4a^3b+2a^4)a}$$

duo valores sompor sunt divorsi, nisi sit vel $b=a$ vel $a=\frac{1}{1\pm\sqrt{3}}$; ill
prodit $q=a=b$, hoc vero reperitur $q=\frac{1-b}{1+2b}$. Punctum ergo curva
d respondet quantitati $\frac{1}{1\pm\sqrt{3}}$, singulari proprietate erit praeditum.

PROBLEMA 3

47. *Invenire integrale completum huius aequationis differentialis*

$$\frac{dy}{\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}}=\frac{2dx}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}}.$$

SOLUTIO

Istud integrale quaositum ex praecodenti problomate colligi potes
iatur enim punctum P in ipso puncto B, ut sit $p=b$, et consideretu
tum punctum A ut fixum, B vero seu P ut variabile, ex quo continu
gnari deboat punctum Q, ut sit arcus AQ duplus arcus AP. Posit
o variabili p loco b sumatur

$$M=\frac{2A+2B(a+p)+C(aa+pp)+2Dap(a+p)+2Eaapp}{(p-a)^2}$$

$$\pm\frac{2}{(p-a)^2}\sqrt{\left(A+2Ba+Caa+2Da^3+Ea^4\right)\left(A+2Bp+Cpp+2Dp^3+Ep^4\right)}$$

ut iam M sit functio variabilis p et constantis a. Deinde posito brev
s gratia $M-C=L$ seu

$$L=\left\{\begin{array}{l}\dfrac{2(A+B(a+p)+Cap+Dap(a+p)+Eaapp)}{(p-a)^2}\\[2ex] \pm\dfrac{2\sqrt{(A+2Ba+Caa+2Da^3+Ea^4)(A+2Bp+Cpp+2Dp^3+Ep^4)}}{(p-a)^2}\end{array}\right\}$$

$$0 = 4AC - 4BB + 4AL + 2(2BL + 4AD)(p+q) + (4A$$
$$+ 2(LL + 2CL + 4AE + 4BD)pq + 2(2DL + 4B$$
$$+ 4(CE - DD + EL)ppqq$$

eritque ob $b = p$

$$\Pi : q - \Pi : p = \Pi : p - \Pi : a \quad \text{seu} \quad \Pi : q = 2\Pi :$$

quae aequatio differentiata dat

$$\frac{dq}{\sqrt{(A + 2Bq + Cqq + 2Dq^3 + Eq^4)}} = \frac{2dp}{\sqrt{(A + 2Bp + Cpp +}}$$

cuius propterea integralis est illa aequatio algebraica int
quam simul patet esse integralem completam, quoniam co
constantem a, quae in aequatione differentiali non inest.

COROLLARIUM 1

48. Si retinente L valorem exhibitum inventaque va
q simili modo quaeratur r, ut sit

$$\Pi : r - \Pi : q = \Pi : p - \Pi : a,$$

erit

$$\Pi : r = 3\Pi : p - 2\Pi : a,$$

unde prodit aequatio differentialis

$$\frac{dr}{\sqrt{(A + 2Br + Crr + 2Dr^3 + Er^4)}} = \frac{3dp}{\sqrt{(A + 2Bp + Cpp +}}$$

cuius ergo aequatio integralis completa est

$$0 = 4(AC - BB + AL) + 2(2BL + 4AD)(q+r) + (4A$$
$$+ 2(LL + 2CL + 4AE + 4BD)qr + 2(2DL + 4B$$
$$+ 4(CE - DD + EL)qqrr.$$

COROLLARIUM 2

...oc magis contrahamus, postquam ex coefficientibus datis A, ... variabili p una cum constanti arbitraria a ita fuerit definita ... sit

$$...-a)^2 = A + B(a+p) + Cap + Dap(a+p) + Eaapp$$

$$...Ba + Caa + 2Da^3 + Ea^4)(A + 2Bp + Cpp + 2Dp^3 + Ep^4),$$

...tur sequentes coefficientes variabiles

$$...-BB+AL),\quad \beta = 2BL + 4AD,\quad \gamma = 4AE - LL,$$

$$...+EL),\quad \varepsilon = 2DL + 4BE,\quad \delta = LL + 2CL + 4AE + 4BD.$$

COROLLARIUM 3

...m quantitatibus inventis erit huius aequationis differentialis

$$\frac{dq}{\ldots q + Cqq + 2Dq^3 + Eq^4)} = \frac{2dp}{\sqrt{(A + 2Bp + Cpp + 2Dp^3 + Ep^4)}}$$

...lis completa

$$\ldots\beta(p+q) + \gamma(pp+qq) + 2\delta pq + 2\varepsilon pq(p+q) + \zeta ppqq.$$

COROLLARIUM 4

...uius aequationis differentialis

$$\frac{dr}{\ldots Br + Crr + 2Dr^3 + Er^4)} = \frac{3dp}{\sqrt{(A + 2Bp + Cpp + 2Dp^3 + Ep^4)}}$$

...lis completa erit

$$\ldots 2\beta(q+r) + \gamma(qq+rr) + 2\delta qr + 2\varepsilon qr(q+r) + \zeta qqrr,$$

...et variabilis q ope praecedentis aequationis ex p fuerit

52. Simili modo progrediendo huius aequationis differentiali

$$\frac{ds}{\sqrt{(A + 2Bs + Css + 2Ds^3 + Es^4)}} = \frac{4dp}{\sqrt{(A + 2Bp + Cpp + 2Dp^3 \cdot}}$$

aequatio integralis completa erit

$$0 = \alpha + 2\beta(r + s) + \gamma(rr + ss) + 2\delta rs + 2\varepsilon rs(r + s) + \zeta$$

postquam ex praecedentibus aequationibus r per q et q per p fuer

COROLLARIUM 6

53. Hoc modo, quousque libuerit, ulterius progredi licet sic aequatio integralis inveniri poterit completa huius differentialis

$$\frac{dx}{\sqrt{(A + 2Bx + Cxx + 2Dx^3 + Ex^4)}} = \frac{mdp}{\sqrt{(A + 2Bp + Cpp + 2Dp^3}}$$

quicunque numerus integer pro m assumatur.

PROBLEMA 4

54. *Si m et n fuerint numeri integri quicunque, invenire aeq gralem completam huius differentialis*

$$\frac{ndy}{\sqrt{(A + 2By + Cyy + 2Dy^3 + Ey^4)}} = \frac{mdx}{\sqrt{(A + 2Bx + Cxx + 2Dx^3}}$$

SOLUTIO

Quaeratur primum ope praeced. probl. aequatio integra istius differentialis

$$\frac{dx}{\sqrt{(A + 2Bx + Cxx + 2Dx^3 + Ex^4)}} = \frac{ndp}{\sqrt{(A + 2Bp + Cpp + 2Dp^3 \cdot}}$$

quae erit algebraica ac praeter variabiles p et x constantem

mo simili modo quaeratur aequatio integralis completa huius

$$\frac{dy}{By + Cyy + 2Dy^3 + Ey^4)} = \frac{mdp}{\sqrt{(A + 2Bp + Cpp + 2Dp^3 + Ep^4)}},$$

algebraica inter binas variabiles y et p insuperque constan-
b complectetur. Ex his duabus aequationibus eliminetur
obtinebitur aequatio algebraica inter x et y, quae erit inte-
huius differentialis

$$\frac{ndy}{By + Cyy + 2Dy^3 + Ey^4)} = \frac{mdx}{\sqrt{(A + 2Bx + Cxx + 2Dx^3 + Ex^4)}}$$

duas constantes arbitrarias a et b continebit, alterutri pro
determinatum tribuere licet vel inter eas datam rationem
ntegrali enim completa sufficit, ut una constans arbitraria

SCHOLION

t n sint numeri modico magni, nemo certe aequationem alge-
et y evolutam exhibebit; cum enim tot eliminationibus sit
st ad aequationem plurimorum terminorum, in qua variabiles
mas dimensiones exsurgant, perveniri oportere. Atque adeo
matis 3, ubi est $m = 2$ et $n = 1$, nemo facile eliminationis
Neque vero hoc etiam opus est, cum ad nostrum institutum
sse aequationem integralem esse algebraicam eiusque con-
metrice absolvi posse; tantum enim abest, ut alienae varia-
., quae in subsidium sunt vocatae, calculum turbent ideoque
nt, ut potius ad constructionem commode instituendam ab-
ssariae.
sunt fere, quae de curvis, quarum rectificatio hac formula

$$\int \frac{\mathfrak{A}dz}{\sqrt{(A + 2Bz + Czz + 2Dz^3 + Ez^4)}}$$

i operae pretium videbatur, quae eo redeunt, ut earum arcus
atque arcus circulares comparari queant, siquidem proposito
AB a puncto dato P arcus abscindi possunt, qui ad illum
t rationalem quamcunque. Consideremus igitur etiam curvas,
tio tali formula exprimitur

$$\int \frac{dz(\mathfrak{A} + \mathfrak{B}z + \mathfrak{C}zz + \mathfrak{D}z^3 + \mathfrak{E}z^4)}{\sqrt{(A + 2Bz + Czz + 2Dz^3 + Ez^4)}},$$

notari merentur; quem in finem evolutio formularum
§ 16 et seqq. est instituta. Similis scilicet comparatio
curvarum suscipi potest, quae iam pridem inter arcu
est ostensa; atque inde sequentium problematum solu

PROBLEMA 5

56. *Proposita curva, cuius arcus indefinite variabi*
hac formula exprimatur

$$\int \frac{dz(\mathfrak{A}+\mathfrak{B}z+\mathfrak{C}zz+\mathfrak{D}z^3+\mathfrak{E}z^4)}{\sqrt{(A+2Bz+Czz+2Dz^3+Ez^4)}}$$

si in ea detur arcus quicunque AB (Fig. 1, p. 341),
abscindere PQ, qui ab illo arcu AB differat linea sive
a circuli hyperbolaeve quadratura pendente.

SOLUTIO

Sit in curva proposita ΔZ arcus variabili z res
gratia ita exprimatur $\Pi : z$, ut sit

$$\Pi : z = \int \frac{dz(\mathfrak{A}+\mathfrak{B}z+\mathfrak{C}zz+\mathfrak{D}z^3+\ldots}{\sqrt{(A+2Bz+Czz+2Dz^3+\ldots}}$$

Punctis autem A, B, P, Q respondeant variabilis z v

$$\Delta A = \Pi : a, \quad \Delta B = \Pi : b, \quad \Delta P = \Pi : p \quad \text{et}$$

hincque erit

arcus datus $AB = \Pi : b - \Pi :$

et

arcus quaesitus $PQ = \Pi : q - \Pi$

Iam primum ex coefficientibus A, B, C, D, E et
deinceps definienda formentur quantitates sequentes

$\alpha = 4(AM - BB)$, $\beta = 2B(M - C) + 4AD$, γ

$\zeta = 4(EM - DD)$, $\varepsilon = 2D(M - C) + 4BE$, $\delta = M$

o statuatur

$$M - C)^2 + 4M(BD - AE) + 4(ADD + BBE) - 4BCD$$

et q haec constituatur relatio, ut sit

$$2\beta(p+q) + \gamma(pp+qq) + 2\delta pq + 2\varepsilon pq(p+q) + \zeta ppqq.$$

riabili p altera q puncto Q respondens ita definitur, ut sit

$$-\beta - \delta p - \varepsilon pp \pm \frac{2\sqrt{\Delta(A + 2Bp + Cpp + 2Dp^3 + Ep^4)}}{\gamma + 2\varepsilon p + \zeta pp},$$

curvae punctum Q, ita ut differentia inter arcus AB et PQ rice assignabilis vel saltem a quadratura circuli seu hyperbolae rei ratio in indole coefficientium $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$ numeratoris nodo igitur differentia ista exprimatur, videamus; quia valorem invenimus, ponamus $p + q = s$ et ex § 19 colligimus fore λ

$$\Pi : q - \Pi : p = \text{Const.} - \frac{2(\mathfrak{D} + \mathfrak{E}s)s\sqrt{\Delta}}{\zeta}$$

$$\frac{\zeta\mathfrak{B} + \lambda\mathfrak{D} + (\zeta\mathfrak{C} + \varepsilon\mathfrak{D} + 2\lambda\mathfrak{E})s + (\zeta\mathfrak{D} + 2\varepsilon\mathfrak{E})ss + \zeta\mathfrak{E}s^3}{\zeta\sqrt{(M + 2Ds + Ess)}}ds,$$

manifestum est vel esse algebraicum vel a quadratura circuli endere. Sit istud integrale brevitatis gratia $= S$; cuius valor b fiat $= I$ et pro constante definienda statuatur $p = a$ et debet

$$\text{nst.} = \Pi : b - \Pi : a + \frac{2(\mathfrak{D} + \mathfrak{E}(a+b))(a+b)\sqrt{\Delta}}{\zeta} - I,$$

ur

$$\text{rcu } AB = \frac{2\mathfrak{D}(a+b) + 2\mathfrak{E}(a+b)^2}{\zeta}\sqrt{\Delta} - \frac{2\mathfrak{D}(p+q) + 2\mathfrak{E}(p+q)^2}{\zeta}\sqrt{\Delta}$$

$$\int \frac{\zeta\mathfrak{B} + \lambda\mathfrak{D} + (\zeta\mathfrak{C} + \varepsilon\mathfrak{D} + 2\lambda\mathfrak{E})s + (\zeta\mathfrak{D} + 2\varepsilon\mathfrak{E})ss + \zeta\mathfrak{E}s^3}{\zeta\sqrt{(M + 2Ds + Ess)}}ds.$$

itraria M etiam ita definiri debet, ut posito $p = a$ fiat $q = b$;

quocirca erit

$$M = \frac{1}{(b-a)^2}\left(2A + 2B(a+b) + C(aa+bb) + 2Dab(a+b\right.$$

$$+ \frac{2}{(b-a)^2}\sqrt{\left(A + 2Ba + Caa + 2Da^3 + Ea^4\right)\left(A + 2Bb + Cbb\right.}$$

Hinc ergo cognita constante hac M et ex puncto P do differentia arcuum AB et PQ vel geometrice vel per qua hyperbolaeve assignari potest.

COROLLARIUM 1

57. Ex datis ergo punctis A et B seu variabilis z primum constans arbitraria M ita definiatur, ut sit

$$M = \frac{1}{(b-a)^2}\left(2A + 2B(a+b) + C(aa+bb) + 2Dab(a+b\right.$$

$$\mp \frac{2}{(b-a)^2}\sqrt{\left(A + 2Ba + Caa + 2Da^3 + Ea^4\right)\left(A + 2Bb + Cbb\right.}$$

Tum hinc definitis modo praecepto coefficientibus α, β, γ, puncto P punctum Q per hanc aequationem determinetur

$$0 = \alpha + 2\beta(p+q) + \gamma(pp+qq) + 2\delta pq + 2\varepsilon pq(p+q$$

atque arcuum PQ et AB differentia erit vel algebraica vel bolaeve quadratura pendens.

COROLLARIUM 2

58. Ad istam autem arcuum differentiam assignandam c $p + q = s$ hoc integrale, ubi $\lambda = \delta - \gamma = 2M(M - C) + 4B$

$$S = \int \frac{\zeta\mathfrak{B} + \lambda\mathfrak{D} + (\zeta\mathfrak{C} + \varepsilon\mathfrak{D} + 2\lambda\mathfrak{E})s + (\zeta\mathfrak{D} + 2\varepsilon\mathfrak{E})ss + \zeta\mathfrak{E}}{\zeta\sqrt{(M + 2Ds + Ess)}}$$

cuius valor posito $s = a + b$ sit $= I$, quo facto erit

$$\text{arc. } PQ - \text{arc. } AB = \frac{2\sqrt{\Delta}}{\zeta}\left(\mathfrak{D}(a+b) + \mathfrak{E}(a+b)^2 - \mathfrak{D}s - \mathfrak{E}\right.$$

existente

$$\Delta = M(M-C)^2 + 4M(BD - AE) + 4(ADD + BBE)$$

COROLLARIUM 3

59. Si eveniret, ut esset $\zeta = 0$, determinatio puncti Q maneret ut [...] pro arcuum PQ et AB differentia assignanda recurri deberet ad [...] operationes. Scilicet ex $p + q = s$ quaeratur t, ut sit

$$0 = \alpha + 2\beta s + \gamma ss + 2\lambda t + 2\varepsilon st + \zeta tt,$$

atque

$$\text{arc. } PQ - \text{arc. } AB = 2\int \frac{ds(\mathfrak{B} + \mathfrak{C}s + \mathfrak{D}(ss - b) + \mathfrak{E}s(ss - 2b))\sqrt{A}}{\sqrt{(\lambda\lambda - \alpha\zeta + 2(\lambda\varepsilon - \beta\zeta)s + (\varepsilon\varepsilon - \gamma\zeta)ss)}}$$

integrali hoc ita accepto, ut evanescat posito $s = a + b$. Ubi notandum est

$$\sqrt{(\lambda\lambda - \alpha\zeta + 2(\lambda\varepsilon - \beta\zeta)s + (\varepsilon\varepsilon - \gamma\zeta)ss)} = 2\sqrt{A(M + 2Ds + Ess)} - \lambda + \varepsilon \ldots$$

COROLLARIUM 4

60. Hinc etiam colligere licet, quaenam sit futura differentia a[rcuum A]B et PQ, si formulae elementum curvae exhibentis numerator ad [...] terminos extendatur, ut sit arcus curvae

$$\int \frac{dz(\mathfrak{A} + \mathfrak{B}z + \mathfrak{C}z^2 + \mathfrak{D}z^3 + \mathfrak{E}z^4 + \mathfrak{F}z^5 + \mathfrak{G}z^6 + \mathfrak{H}z^7 + \text{etc.})}{\sqrt{(A + 2Bz + Czz + 2Dz^3 + Ez^4)}};$$

[rel]iquis enim manentibus ut ante erit

$$\text{arc. } PQ - \text{arc. } AB = \int \frac{ds(\mathfrak{B} + \mathfrak{C}s + \mathfrak{D}(ss - b) + \mathfrak{E}(s^3 - 2sb) + \mathfrak{F}(s^4 - 3sst + tt) \ldots}{\sqrt{(M + 2Ds + Ess)}}$$

[seq]uentia scilicet numeratoris membra erunt

$$\mathfrak{G}(s^5 - 4s^3t + 3stt) + \mathfrak{H}(s^6 - 5s^4t + 6sstt - t^3) + \text{etc.}$$

COROLLARIUM 5

61. Si a puncto Q simili modo abscindatur R, ut sit

$$0 = \alpha + 2\beta(q + r) + \gamma(qq + rr) + 2\delta qr + 2\varepsilon qr(q + r) + \zeta qqrr,$$

[p]onaturque $q + r = u$ et $qr = v$, ita ut sit

$$0 = \alpha + 2\beta u + \gamma uu + 2\lambda v + 2\varepsilon uv + \zeta vv$$

[eri]t

$$\lambda + \varepsilon u + \zeta v = 2\sqrt{A(M + 2Du + Euu)},$$

$$\text{arc. } PR - 2\,\text{arc. } AB = \int \frac{ds(\mathfrak{B} + \mathfrak{C}s + \mathfrak{D}(ss-t) + \mathfrak{E}(s^3 - 2s}{\sqrt{(M + 2Ds + Ess)}}$$

$$+ \int \frac{du(\mathfrak{B} + \mathfrak{C}u + \mathfrak{D}(uu - v) + \mathfrak{E}(u^3 - 2uv) + \text{etc.})}{\sqrt{(M + 2Du + Euu)}}$$

his integralibus ita sumtis, ut evanescant posito $s = a + b$ et

COROLLARIUM 6

62. Simili modo a puncto P abscindi potest arcus PS, q
arcus AB superet quantitate sive geometrice assignabili sive
bolaeve quadratura pendente, hisque casibus punctum P ita
ut iste excessus plane evanescat, quod quidem semper prae
excessus sit algebraicus; sin autem sit transcendens, insuper
arcus dati A vel B huic scopo conformiter determinabitur.

OVA SERIES INFINITA MAXIME CONVERGENS PERIMETRUM ELLIPSIS EXPRIMENS

Commentatio 448 indicis ENESTROEMIANI
ovi commentarii academiae scientiarum Petropolitanae 18 (1773), 1774, p. 71—84
Summarium ibidem p. 13—15

SUMMARIUM

n Commentariis Academiae nostrae uti et in Actis Berolinensibus passim iam ctor series dedit infinitas, quibus ellipsis cuiuscunque perimeter exprimitur, tam as et simplices, ut dari alias adhuc commodiores vix suspicari licuerit. Haec series, quam Ill. Auctor in praesenti dissertatione proponit, ceteris concinnitate sua enda videtur estque plane nova. Quadrantis elliptici ponantur semiaxes a et b parallelae coordinatae x et y; habebitur ex natura ellipsis

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

aequatione Ill. Auctor peringeniose totius arcus seu quartae partis perimetri longi m determinat. Ponatur scilicet

$$x=a\sqrt{\frac{1+z}{2}} \quad \text{et} \quad y=b\sqrt{\frac{1-z}{2}},$$

$$dx=\frac{adz}{2\sqrt{2(1+z)}} \quad \text{et} \quad dy=\frac{-bdz}{2\sqrt{2(1-z)}};$$

o, si arcus ponatur $=s$, habebitur

$$ds^2=dz^2\frac{a^2+b^2-(a^2-b^2)z}{8(1-z^2)}$$

ne

$$s=\frac{1}{2\sqrt{2}}\int dz\sqrt{\frac{a^2+b^2-(a^2-b^2)z}{1-z^2}};$$

aque hoc integrale ita sumatur, ut posito $x=0$ evanescat, et usque ad terminu

$x = a$ extendatur, ...
tialis evolutione III. Auctor versatur ex eaque seriem hanc simplicem
gentem elicit

$$s = \frac{c\pi}{2\sqrt{2}}\left(1 - \frac{1\cdot 1}{4\cdot 4}n^2 - \frac{1\cdot 1}{4\cdot 4}\cdot\frac{3\cdot 5}{8\cdot 8}n^4 - \frac{1\cdot 1}{4\cdot 4}\cdot\frac{3\cdot 5}{8\cdot 8}\cdot\frac{7\cdot 9}{12\cdot 12}n^6\right.$$

ubi

$$c = \sqrt{(a^2 + b^2)} \quad \text{et} \quad n = \frac{a^2 - b^2}{a^2 + b^2}.$$

Si sit $a = b$, quadrans hic ellipticus in circularem abit et ob $n =$
uti quidem notissimum est, $s = \frac{a\pi}{2}$. Si vero ponatur $b = 0$, curva
alteri semiaxi aequalem; ita autem est $n = 1$ et $c = a$; unde sequens

$$a = \frac{a\pi}{2\sqrt{2}}\left(1 - \frac{1\cdot 1}{4\cdot 4} - \frac{1\cdot 1}{4\cdot 4}\cdot\frac{3\cdot 5}{8\cdot 8}\text{ etc.}\right)$$

adeoque seriei infinitae

$$1 - \frac{1\cdot 1}{4\cdot 4} - \frac{1\cdot 1}{4\cdot 4}\cdot\frac{3\cdot 5}{8\cdot 8}\text{ etc.},$$

quae quidem minime convergit, adcurate assignari potest summa $\frac{2\sqrt{2}}{\pi}$
III. Auctor operae pretium censet in summam huius seriei etiam a
quod praestandum methodo sua iam saepius explicata potissimum
quaestionem ad aequationem differentialem revocat, cuius integrale
positam exprimatur.

1. Postquam olim multum fuissem occupatus, ut pl
quibus cuiusque ellipsis perimeter exprimeretur, investiga
spicatus adhuc simpliciores atque ad calculum magis a
modi series erui posse, quam passim dedi sive in Comm
in Actis Berolin.[2])

2. Nunc autem cum forte cogitationes meae in idem
rent, alia ac, ni fallor, multo simplicior et commodior se
cuius investigationem ita animo institui.

Considero scilicet quadrantem ellipticum ACB (Fi
semiaxes sint $CA = a$, $CB = b$, quibus coordinatae

---

1) L. Euleri Commentatio 52 (indicis Enestroemiani); vide p. 8
2) L. Euleri Commentatio 154 (indicis Enestroemiani); vide p.

t $PM = y$, ita ut ex natura ellipsis habeatur
tio

$$bbx^2 + aay^2 = aa \cdot bb$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

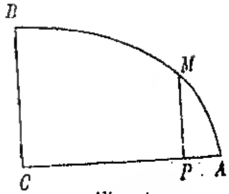


Fig. 1.

ngulari modo definio longitudinem totius arcus AMB sive quartae rimetri.

Cum igitur esse debeat

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

ariabilem z in calculum introduco statuendo

$$\frac{x^2}{a^2} = \frac{1+z}{2},$$

$$\frac{y^2}{b^2} = \frac{1-z}{2},$$

odit

$$x = a\sqrt{\frac{1+z}{2}} \quad \text{et} \quad y = b\sqrt{\frac{1-z}{2}}$$

differentiando

$$dx = \frac{a\,dz}{2\sqrt{2(1+z)}} \quad \text{et} \quad dy = \frac{-b\,dz}{2\sqrt{2(1-z)}};$$

, si vocemus arcum $BM = s$, statim colligimus

$$ds^2 = dx^2 + dy^2 = \frac{a^2dz^2}{8(1+z)} + \frac{b^2dz^2}{8(1-z)}$$

$$ds^2 = \frac{dz^2}{8}\left(\frac{a^2}{1+z} + \frac{b^2}{1-z}\right) = \frac{dz^2(a^2+b^2-(a^2-b^2)z)}{8(1-z^2)}$$

e integrando

$$s = \frac{1}{2\sqrt{2}}\int dz\sqrt{\frac{a^2+b^2-(a^2-b^2)z}{1-z^2}}$$

ali ita sumto, ut evanescat posito $x = 0$ sive $z = -1$; tum vero inte- extendatur usquo ad terminum $x = a$, ubi fit $z = +1$, sicque obtinebitur itus quadrans ellipticus AMB.

$$a^2 + b^2 = c^2 \quad \text{et} \quad \frac{a^2 - b^2}{a^2 + b^2} = n.$$

Hoc enim modo consequimur

$$s = \frac{c}{2\sqrt{2}} \int dz \frac{\sqrt{(1 - nz)}}{\sqrt{(1 - z^2)}},$$

ubi superius radicale more solito in seriem convertamus

$$\sqrt{(1 - nz)} = 1 - \frac{1}{2} nz - \frac{1 \cdot 1}{2 \cdot 4} n^2 z^2 - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} n^3 z^3 - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} n^4 z^4 - \frac{1 \cdot 1 \cdot}{2 \cdot 4 \cdot}$$

qui singuli termini nos ad singulares integrationes perducunt;
priores secundum legem datam integrati, ut scilicet evanescant su
dabunt

$$\int \frac{dz}{\sqrt{(1 - z^2)}} = \text{A. sin. } z - \text{A. sin. } (-1) = \text{A. sin. } z + \frac{1}{2}$$

$$\int \frac{z\,dz}{\sqrt{(1 - z^2)}} = -\sqrt{(1 - z^2)} + 0;$$

hinc ergo, si sumamus $z = +1$, prodibit

$$\int \frac{dz}{\sqrt{(1 - z^2)}} = \pi \quad \text{et} \quad \int \frac{z\,dz}{\sqrt{(1 - z^2)}} = 0.$$

5. Pro reliquis terminis consideremus reductionem consue

$$\int \frac{z^{\lambda+2} dz}{\sqrt{(1 - z^2)}} = A \cdot \int \frac{z^\lambda dz}{\sqrt{(1 - z^2)}} + B \cdot z^{\lambda+1} \sqrt{(1 - z^2)},$$

ubi esse oportet

$$A = \frac{\lambda + 1}{\lambda + 2} \quad \text{et} \quad B = \frac{-1}{\lambda + 2},$$

ita ut sit

$$\int \frac{z^{\lambda+2} dz}{\sqrt{(1 - z^2)}} = \frac{\lambda + 1}{\lambda + 2} \int \frac{z^\lambda dz}{\sqrt{(1 - z^2)}} - \frac{1}{\lambda + 2} z^{\lambda+1} \sqrt{(1 - z^2}$$

ubi constantem non adiicimus, quia haec formula iam e

unde, si iam ponatur $z = +1$, obtinebitur

$$\int \frac{z^{\lambda+2}dz}{\sqrt{(1-z^2)}} = \frac{\lambda+1}{\lambda+2}\int \frac{z^{\lambda}dz}{\sqrt{(1-z^2)}}.$$

x hac reductione statim liquet omnia integralia ex potestatibus ipsius z oriunda per se evanescere; pro potestatibus autem paribus n nostrum adipiscimur

$$\int \frac{dz}{\sqrt{(1-z^2)}} = \pi, \quad \int \frac{z^2dz}{\sqrt{(1-z^2)}} = \frac{1}{2}\pi,$$

$$\int \frac{z^4dz}{\sqrt{(1-z^2)}} = \frac{1\cdot 3}{2\cdot 4}\pi, \quad \int \frac{z^6dz}{\sqrt{(1-z^2)}} = \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\pi$$

etc.

His igitur valoribus substitutis longitudo quadrantis elliptici colli-
re

$$AMB = \frac{c\pi}{2\sqrt{2}}\left\{\begin{aligned} &1 - \frac{1\cdot 1}{2\cdot 4}n^2\cdot\frac{1}{2} - \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}n^4\cdot\frac{1\cdot 3}{2\cdot 4}\\ &- \frac{1\cdot 1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12}n^6\cdot\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\ \text{etc.}\end{aligned}\right\}$$

autem forma scribamus tantisper brevitatis gratia

$$AMB = \frac{c\pi}{2\sqrt{2}}\left(1 - \alpha n^2 - \beta n^4 - \gamma n^6 - \delta n^8 - \varepsilon n^{10}\ \text{etc.}\right),$$

fficientes sequenti modo succinctius exprimi poterunt

$$\alpha = \frac{1\cdot 1}{2\cdot 4}\cdot\frac{1}{2} = \frac{1\cdot 1}{4\cdot 4}, \quad \frac{\beta}{\alpha} = \frac{3\cdot 5}{8\cdot 8}, \quad \frac{\gamma}{\beta} = \frac{7\cdot 9}{12\cdot 12}, \quad \frac{\delta}{\gamma} = \frac{11\cdot 13}{16\cdot 16} \quad \text{etc.}$$

Cum igitur inventi coefficientes tam simplicem et egregiam constituant , haec expressio, quam eruimus, utique maxime videtur attentione digna, ermini vehementer convergant idque pro omnibus plane ellipsibus, prop-

terea quod semper $\frac{}{a^2+b^2}$ ...

$$AMB = \frac{c\pi}{2\sqrt{2}} \left\{ \begin{array}{l} 1 - \frac{1\cdot 1}{4\cdot 4}n^2 - \frac{1\cdot 1}{4\cdot 4}\cdot\frac{3\cdot 5}{8\cdot 8}n^4 - \frac{1\cdot 1}{4\cdot 4}\cdot\frac{3\cdot 5}{8\cdot 8}\cdot\frac{7\cdot 9}{12\cdot 1\ldots} \\ \qquad - \frac{1\cdot 1}{4\cdot 4}\cdot\frac{3\cdot 5}{8\cdot 8}\cdot\frac{7\cdot 9}{12\cdot 12}\cdot\frac{11\cdot 13}{16\cdot 16}n^8 \text{ etc.} \end{array} \right.$$

9. Contemplemur hinc casum, quo ellipsis nostra fit circu... tum enim erit $b = a$, hinc $c = a\sqrt{2}$ et $n = 0$, ex quo quad... prodit, uti quidem notissimum est, $= \frac{1}{2}\pi a$.

10. Deinde vero etiam casus occurrit maximo notatu dignu... $CB = b = 0$; tum enim quadrans ellipticus AMB ipsi semia... aequalis; at pro nostra formula erit $c = a$ et $n = 1$, quibus v... tutis nanciscimur sequentem aequationem

$$a = \frac{\pi a}{2\sqrt{2}}\left(1 - \frac{1\cdot 1}{4\cdot 4} - \frac{1\cdot 1}{4\cdot 4}\cdot\frac{3\cdot 5}{8\cdot 8} - \frac{1\cdot 1}{4\cdot 4}\cdot\frac{3\cdot 5}{8\cdot 8}\cdot\frac{7\cdot 9}{12\cdot 12} \text{ et...}\right.$$

qui praecise ipse ille casus est, quo series nostra quam mini... gens, et qui propterea nostram attentionem eo magis moret... seriei summa adcurate assignari potest, cum sit

$$1 - \frac{1\cdot 1}{4\cdot 4} - \frac{1\cdot 1}{4\cdot 4}\cdot\frac{3\cdot 5}{8\cdot 8} \text{ etc. in infin.} = \frac{2\sqrt{2}}{\pi}.$$

10[a][1]. Si cui lubuerit super hac serie calculos numericos... iungamus hic valores litterarum α, β, γ etc. in fractionib... qui ita se habent

$$\begin{aligned} \alpha &= 0{,}0625000 \\ \beta &= 0{,}0146484 \\ \gamma &= 0{,}0064087 \\ \delta &= 0{,}0035798 \\ \varepsilon &= 0{,}0022821 \\ \zeta &= 0{,}0015808 \\ &\text{etc.,} \end{aligned}$$

1) In editione principe falso numerus 10 iteratur. A. K.

m hucusque tantum continuata prodit

$$1-\alpha-\beta-\gamma-\delta-\varepsilon-\zeta=0{,}9090002;$$

reperitur $\frac{2\sqrt{2}}{\pi}=0{,}9003200$; unde videmus sequentium litterarum etc. omnium summam efficere debere 0,0086802.

Ceterum pro calculo numerico non parum notasse iuvabit nostros tes etiam sequenti modo concinnius exprimi posse

$$\alpha=\frac{1}{16}$$

$$\beta=\frac{1}{64}\cdot\frac{15}{16}$$

$$\gamma=\frac{1}{144}\cdot\frac{15}{16}\cdot\frac{63}{64}$$

$$\delta=\frac{1}{256}\cdot\frac{15}{16}\cdot\frac{63}{64}\cdot\frac{143}{144}$$

$$\varepsilon=\frac{1}{400}\cdot\frac{15}{16}\cdot\frac{63}{64}\cdot\frac{143}{144}\cdot\frac{255}{256}$$

etc.

2. Occasione huius seriei, quam invenimus, operae pretium erit in eius am a posteriore inquirere, id quod duplici modo fieri potest; prior , quem iam olim[1] proposui ac deinceps saepissime ad usum accommo- nos deducit ad aequationem differentialem, cuius integrale per ipsam propositam exprimatur. Quo nunc haec methodus facilius adhiberi ponamus $n=2v$, ut series summanda fiat

$$s=1-\frac{1\cdot1}{2\cdot2}v^2-\frac{1\cdot1}{2\cdot2}\cdot\frac{3\cdot5}{4\cdot4}v^4-$$

1) L. Euleri Commentatio 19 (indicis Ene *arum termini generales algebraice dari neq* p. 36; *Leonhardi Euleri Opera omnia*, se

ut prodeat

$$\frac{vds}{dv} = -\frac{1\cdot 1}{2}v^2 - \frac{1\cdot 1}{2\cdot 2}\cdot\frac{3\cdot 5}{4}v^4 - \frac{1\cdot 1}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{7\cdot 9}{6}v^6 \text{ etc.},$$

quae denuo differentiata praebet

$$\frac{d.vds}{dv^2} = -1\cdot 1v - \frac{1\cdot 1}{2\cdot 2}3\cdot 5v^3 - \frac{1\cdot 1}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}7\cdot 9v^5 \text{ etc.};$$

hoc scilicet modo ex singulis denominatoribus duos factores sustul

14. Nunc vero denuo ope differentiationis numeratores binis ribus augeamus; hunc in finem primam aequationem in $\sqrt{v}$ ductam tiemus prodibitque

$$\frac{2d.s\sqrt{v}}{dv} = +v^{-\frac{1}{2}} - \frac{1\cdot 1}{2\cdot 2}5v^{\frac{3}{2}} - \frac{1\cdot 1}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}9v^{\frac{7}{2}} - \frac{1\cdot 1}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{7\cdot 9}{6\cdot 6}13v$$

haec denuo differentietur et per 2 iterum multiplicando fit

$$\frac{4dd.s\sqrt{v}}{dv^2} = -v^{-\frac{3}{2}} - \frac{1\cdot 1}{2\cdot 2}3\cdot 5v^{\frac{1}{2}} - \frac{1\cdot 1}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}7\cdot 9v^{\frac{5}{2}} \text{ etc.},$$

quae per $v^{\frac{5}{2}}$ multiplicata producit

$$\frac{4v^{\frac{5}{2}}dd.s\sqrt{v}}{dv^2} = -v - \frac{1\cdot 1}{2\cdot 2}3\cdot 5v^3 - \frac{1\cdot 1}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}7\cdot 9v^5 \text{ etc.};$$

supra vero iam invenimus

$$\frac{d.vds}{dv^2} = -v - \frac{1\cdot 1}{2\cdot 2}3\cdot 5v^3 - \frac{1\cdot 1}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}7\cdot 9v^5 \text{ etc.},$$

quae series cum sint aequales, inde deducimus hanc aequationem

$$4v^{\frac{5}{2}}dd.s\sqrt{v} = d.vds,$$

quae aequatio continet relationem summae quaesitae s ad variabi

Haec ergo aequatio evoluta fit differentialis secundi gradus; sumt
mento dv constante ob

$$d.s\sqrt{v} = ds\sqrt{v} + \frac{sdv}{2\sqrt{v}}$$

$$dd.s\sqrt{v} = dds\cdot\sqrt{v} + \frac{dvds}{\sqrt{v}} - \frac{sdv^2}{4v\sqrt{v}},$$

$$4v^2 dd.s\sqrt{v} = 4v^3dds + 4v^2dvds - svdv^2;$$

ro ob $d.vds = vdds + dvds$ habebitur haec aequatio

$$vdds(1-4v^2) + dvds(1-4v^2) + svdv^2 = 0$$

$$vdds + dvds + \frac{svdv^2}{1-4v^2} = 0.$$

. Huius igitur aequationis differentialis secundi gradus constructi
est potestate; fiat enim ellipsis, cuius semiaxes sint a et b eiu
riae quarta pars $= q = AMB$; tum vero capiatur

$$c = \sqrt{(a^2+b^2)} \text{ et } \frac{a^2-b^2}{a^2+b^2} = n = 2v;$$

cum sit

$$q = \frac{\pi c}{2\sqrt{2}}s,$$

$$s = \frac{2q\sqrt{2}}{\pi c}.$$

ob $a^2+b^2 = c^2$ et $a^2-b^2 = 2c^2v$ erit

$$a^2 = \frac{c^2(1+2v)}{2} \quad\text{et}\quad b^2 = \frac{c^2(1-2v)}{2}.$$

irca nostra constructio ita erit comparata: sumtis ellipsis semiaxib

$$a = c\sqrt{\frac{1+2v}{2}} \quad\text{et}\quad b = c\sqrt{\frac{1-2v}{2}}$$

q quarta pars perimetri huius ellipsis eritque pro resolutione
ationis $s = \frac{2q\sqrt{2}}{\pi c}$.

$$ddz + \frac{z\,dv^2}{4v^2(1-4v^2)} = 0,$$

pro qua erit

$$z = \frac{2q\sqrt{2v}}{\pi c}.$$

17. Haec porro aequatio ad differentialem primi gradus
nendo $z = e^{\int t\,dv}$; tum enim resultabit

$$dt + t^2 dv + \frac{dv}{4v^2(1-4v^2)} = 0,$$

unde si liceret t per v definire, ita ut innotesceret integral
$z = e^{\int t\,dv}$.

18. Hic erat primus modus ex proposita serie infinita in
inquirendi, ubi scilicet loco numeri constantis n quantitatem va
troduximus; altero autem modo idem praestandi, cuius plur
iam passim occurrunt, quantitas constans n talis relinquitur; p
$n = 2m$, ita ut nostra series summanda sit

$$1 - \frac{1\cdot 1}{2\cdot 2}m^2 - \frac{1\cdot 1}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}m^4 - \frac{1\cdot 1}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{7\cdot 9}{6\cdot 6}m^6 \text{ etc.}$$

19. Nunc fingamus esse

$$s = \int dz\sqrt[4]{(1-2m^2p)},$$

postquam scilicet absoluta integratione quantitati variabili z
determinatus fuerit tributus; litteram vero p etiam ut variabil
quae cuiusmodi functio ipsius z capi debeat, ut haec integrati
seriem infinitam perducatur, sequenti modo investigabimus.

20. Evoluta autem formula irrationali $(1-2m^2p)^{\frac{1}{4}}$ in har
nitam

$$1 - \frac{1}{2}m^2p - \frac{1\cdot 3}{2\cdot 4}m^4p^2 - \frac{1\cdot 3\cdot 7}{2\cdot 4\cdot 6}m^6p^3 \text{ etc.}$$

e sequenti serie formularum integralium definietur

$$s = z - \frac{1}{2}m^2\int p\,dz - \frac{1\cdot3}{2\cdot4}m^4\int p^2dz - \frac{1\cdot3\cdot7}{2\cdot4\cdot6}m^6\int p^3dz \text{ etc.}$$

ro statuamus, si post singulas integrationes variabili z certus valo
atus tribuatur, tum fore

$$\int p\,dz = \frac{1}{2}z,\quad \int p^2dz = \frac{5}{4}\int p\,dz,$$

$$\int p^3dz = \frac{9}{6}\int p^2dz,\quad \int p^4dz = \frac{13}{8}\int p^3dz$$

etc.;

fiet

$$s = z\left(1 - \frac{1\cdot1}{2\cdot2}m^2 - \frac{1\cdot1}{2\cdot2}\cdot\frac{3\cdot5}{4\cdot4}m^4 - \frac{1\cdot1}{2\cdot2}\cdot\frac{3\cdot5}{4\cdot4}\cdot\frac{7\cdot9}{6\cdot6}m^6 \text{ etc.}\right),$$

st ipsa nostra series proposita.

. Nunc igitur tota quaestio huc redit, cuiusmodi functionem ipsius z p
oporteat, ut stabilita illa ratio integralium, dum scilicet variabil
valor tribuitur, obtineatur; ista autem relatio generatim ita exprimi

$$\int p^\lambda dz = \frac{4\lambda-3}{2\lambda}\int p^{\lambda-1}dz;$$

us igitur integralibus adhuc indefinite sumtis fore

$$\int p^\lambda dz = \frac{4\lambda-3}{2\lambda}\int p^{\lambda-1}dz + \frac{p^\lambda Q}{2\lambda};$$

ergo differentiatione prodibit

$$p^\lambda dz = \frac{4\lambda-3}{2\lambda}p^{\lambda-1}dz + \frac{1}{2}p^{\lambda-1}Q\,dp + \frac{p^\lambda}{2\lambda}dQ,$$

per $p^{\lambda-1}$ divisa et per 2λ multiplicata praebet

$$2\lambda p\,dz = (4\lambda-3)dz + \lambda Q\,dp + p\,dQ,$$

um haec aequatio subsistere debeat, quicquid sit λ, suppeditat nobis

$$2pdz - 4dz - Qdp = 0, \quad -3dz + pdQ = 0,$$

ex quibus utramque functionem p et Q definire licebit.

22. Perindo autem hic est, sive p et Q sint functiones ipsius
et Q ipsius p, dummodo earum relatio inter se stabiliatur; ex
autem statim habemus

$$dz = \frac{1}{3} p dQ,$$

qui valor in priore substitutus praebet

$$\frac{2}{3}(p-2)pdQ - Qdp = 0,$$

ex qua fit

$$\frac{dQ}{Q} = \frac{3dp}{2p(p-2)} = -\frac{3dp}{4p} + \frac{3dp}{4(p-2)},$$

unde integrando oritur

$$\log. Q = -\frac{3}{4}\log. p + \frac{3}{4}\log. (p-2) = +\frac{3}{4}\log. \frac{p-2}{p},$$

unde fit

$$Q = 2\left(\frac{p-2}{p}\right)^{\frac{3}{4}};$$

tum vero, quia ex prima aequatione est $dz = \frac{Qdp}{2(p-2)}$, hinc fit

$$dz = \frac{dp}{p^{\frac{3}{4}}(p-2)^{\frac{1}{4}}} = \frac{dp}{\sqrt[4]{p^3(p-2)}}.$$

Nunc autem inprimis observari oportet, ut pro utroque integration
formula algebraica ibi adiecta

$$p^\lambda Q = 2p^{\lambda - \frac{3}{4}}(p-2)^{\frac{3}{4}}$$

evanescat, sicque manifestum est integrationis terminos statui de
et $p = 2$.

23. Ecce ergo formulam nostram integralem initio introdu
modo repraesentatam

$$s = \int \frac{dp\sqrt[4]{(1-2m^2p)}}{\sqrt[4]{p^3(p-2)}};$$

$$z=\int\frac{dp}{\sqrt[4]{p^3(p-2)}},$$

series proposita

$$1-\frac{1\cdot 1}{2\cdot 2}m^2-\frac{1\cdot 1}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}m^4 \text{ etc.}$$

fractioni $\frac{s}{z}$, postquam scilicet haec integralia ita fuerint sumta, cant posito $p=0$, tum vero statuatur $p=2$; quamobrem illas duas integrales ita exprimi conveniet

$$s=\int\frac{dp\sqrt[4]{(1-2m^2p)}}{\sqrt[4]{p^3(2-p)}} \quad \text{et} \quad z=\int\frac{dp}{\sqrt[4]{p^3(2-p)}}.$$

Ex his igitur series nostra supra inventa

$$1-\frac{1\cdot 1}{4\cdot 4}n^2-\frac{1\cdot 1}{4\cdot 4}\cdot\frac{3\cdot 5}{8\cdot 8}n^4 \text{ etc.},$$

mmam iam vidimus esse $\frac{2q\sqrt{2}}{\pi c}$, etiam hoc modo per duas formulas s repraesentari potest, quae facta levi mutatione $p=2r$ erunt, ea, meratorem constituit,

$$s=\int\frac{dr\sqrt[4]{(1-nnr)}}{\sqrt[4]{r^3(1-r)}},$$

ero, quae constituit denominatorem,

$$z=\int\frac{dr}{\sqrt[4]{r^3(1-r)}};$$

tem fractio nostram seriem exhibebit; nunc autem termini integrationis et $r=1$.

5. Adhuc succinctius hae formulae transformari possunt sumendo tum enim ambae formulae integrales erunt

$$s=\int\frac{dt\sqrt[4]{(1-n^2t^4)}}{\sqrt[4]{(1-t^4)}} \quad \text{et} \quad z=\int\frac{dt}{\sqrt[4]{(1-t^4)}}$$

terminis integrationis existentibus

fractio $\frac{s}{z}$ aequabitur nostrae seriei sive erit

$$\frac{s}{z}=\frac{2q\sqrt{2}}{\pi c},$$

ubi q denotat quartam partem peripheriae ellipsis, cuius semiax

$$c\sqrt{\frac{1+n}{2}} \quad \text{et} \quad c\sqrt{\frac{1-n}{2}}.$$

26. Hinc casu $n=0$ manifesto fit $\frac{s}{z}=1$, casu vero $n=1$

fiet

$$\frac{1}{z}=\frac{2\sqrt{2}}{\pi} \quad \text{sive} \quad z=\int\frac{dt}{\sqrt{(1-t^4)}}=\frac{\pi}{2\sqrt{2}},$$

quod quidem iam aliunde constat.

SUMMARIUM

Commentationis 28 indicis ENESTROEMIANI

SPECIMEN DE CONSTRUCTIONE AEQUATIONUM DIFFERENTIALIUM SINE INDETERMINATARUM SEPARATIONE[1])

Ex manuscriptis academiae scientiarum Petropolitanae nunc primum editum[2])

Quotiescunque in resolvendo problemate ad aequationem differentialem perventum e
cesse est ad plenariam eius solutionem, ut ista aequatio integretur aut saltem geometri
nstruatur. At *neque integratio neque constructio geometrica facile succedunt, nisi quan*
quatio eo sit perducta, ut litterae variabiles seu indeterminatae in quolibet termino aequ
nis ab invicem seiunctae sint. Hanc ob causam separatio indeterminatarum res maxi
menti est in rebus analyticis. Extant quidem passim methodi particulares integrandi a
nstruendi *aequationes differentiales absque indeterminatarum separatione.* Observavit aute
EULERUS iis solum casibus eas methodos succedere, ubi indeterminatarum separatio a
ilis sit aut ex ipsa constructione elici possit. Ut igitur hanc rem magis perficeret, exe
m adducit aequationis, in qua indeterminatae *nullo* modo separari possunt, atque hui
di *aequationis constructionem tradit geometricam ope rectificationis ellipsis.*

1) Vide p. 1. A. K.

2) Vide p. X praefationis. A. K.